DECAY LAW OF PARTICLES IN FLIGHT REVISITED

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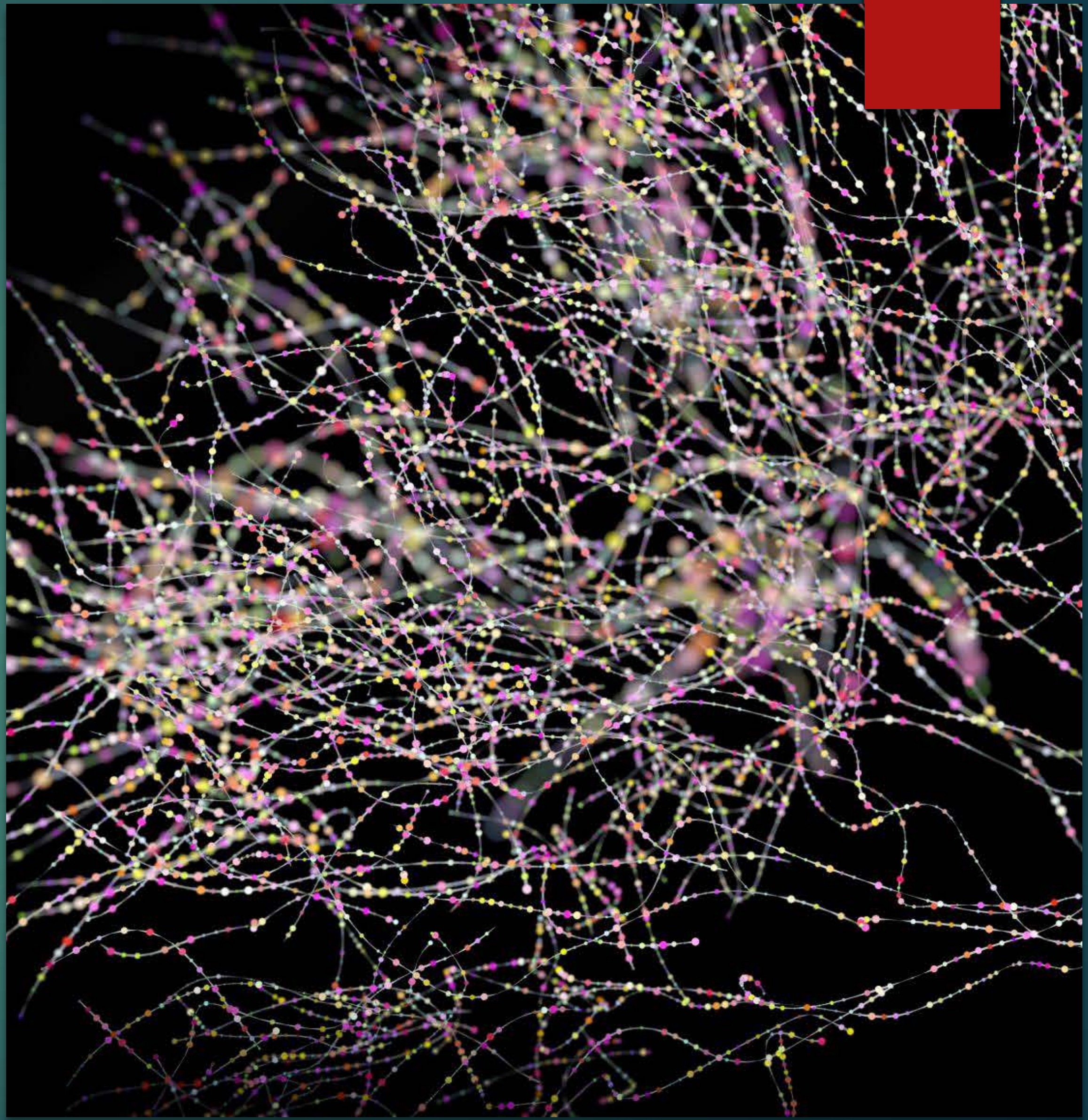


Exponential decay law at rest is essentially classical

- In flight we invoke Special

Relativity: time dilation for lifetime

Tested experimentally up to a certain accuracy (atmospheric muons)

$$\Delta N \propto \Delta t N \rightarrow \frac{dN}{dt} = -\Gamma N$$

$$\rightarrow N(t) = N_0 e^{-\Gamma t} \rightarrow P(t) = \frac{N}{N_0} = e^{-\Gamma t}$$

Lifetime at rest

$$\tau = 1/\Gamma$$

In flight

$$P(t) = e^{-\Gamma t/\gamma}$$

Enters Quantum Mechanics

Fock-Krylov method as a theoretical framework: Let $|\Psi\rangle$ be the unstable state at $t = 0$, then

$$H|\Psi\rangle \neq E|\Psi\rangle$$

otherwise

$$A(t) = \langle\Psi|e^{-iHt}|\Psi\rangle = e^{-iEt} \Rightarrow P(t) = 1$$

$$P(t) = |A(t)|^2$$

Hence, assuming a continuum $H|E\rangle = E|E\rangle$, $\langle E'|E\rangle = \delta(E' - E)$ we get

$$|\Psi\rangle = \int_{\mathrm{Spect(H)}} dEa(E)|E\rangle$$

$$\rho(E) \equiv \frac{\mathrm{Prob}_\Psi(E)}{dE} = |\langle E|\Psi\rangle|^2 = |a(E)|^2$$

$$A(t) = \int_{\mathrm{Spect(H)}} dE\rho(E)e^{-iEt} = \int_{E_{th}}^{\infty} \rho(E)e^{-iEt}$$

General choice of the spectral function

$$\rho(E) = (\text{Threshold}) \times (\text{Pole}) \times (\text{Form} - \text{factor})$$

$$\rho(E) = (E - E_{th})^{\gamma} \times P(E) \times F(E)$$

- $P(E)$: has a simple pole $z_R = E_R - i\Gamma_R/2$ (leads to exponential decay law) corresponding to one resonance. More poles in the fourth quadrant of the complex E-plane would modify even the exponential part of the decay.
- $F(E)$: no threshold, no poles; smooth function which should go to zero for large E.
- Large t corresponds in the Fourier transform to small E. Hence, large time behaviour is due to the choice of γ.

Calculation of the integral in the complex plane

$$e^{-iE_{th}t} I \equiv e^{-iE_{th}t} \left(\int_{C_{Re}} \dots + \int_{C_R} \dots + \int_{C_{Im}} \dots \right)$$

$$A(t) = A_E(t) + A_{LT}(t)$$

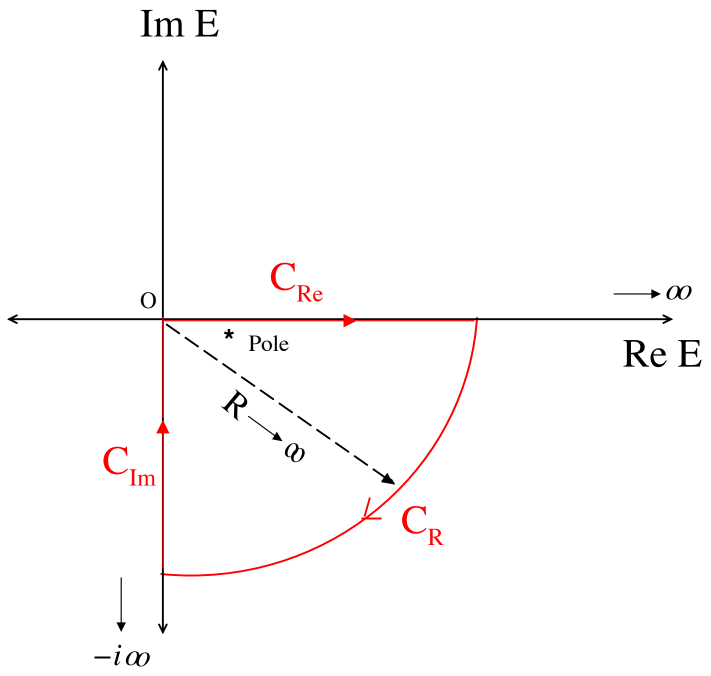
$$A_E(t) = 2\pi i \tilde{P}(z_R) F(z_R)(z_R - E_{th})^{\gamma} e^{iE_R t} e^{-\Gamma_R t/2}$$

$$A_E(t) = a_E(t) e^{-\Gamma_R t/2}$$

$$\tilde{P}(z) = P(z)(z - z_R)$$

For simplicity the small time is left out

Residue Theorem gives exponential law



Large time behavior corresponds to integral along the imaginary axis

$$A_{LT}(t) = (\text{phase}) \times \int_0^\infty dx P(-ix+E_{th}) F(-ix+E_{th}) x^\gamma e^{-xt}$$

For large times, corresponding to small x

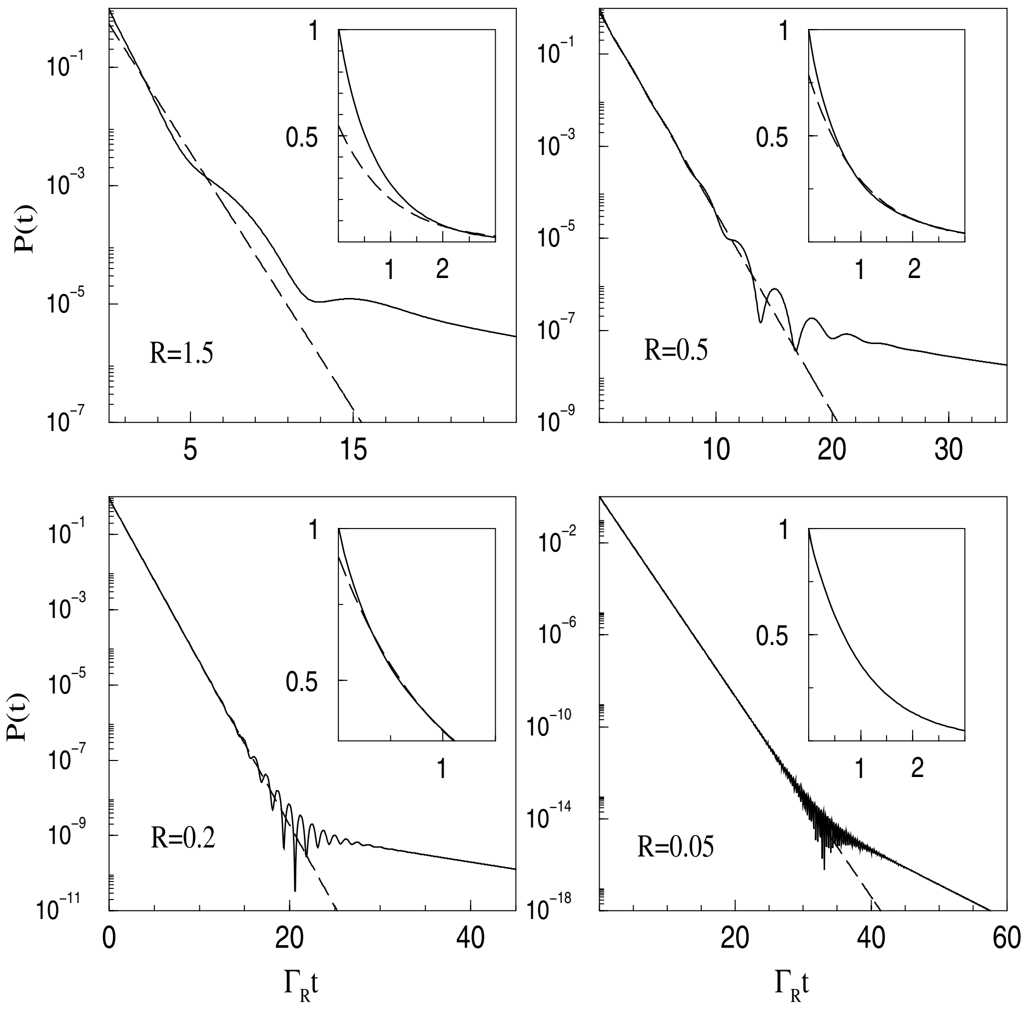
$$A_{LT}(t) \simeq (\text{phase}) \times P(E_{th}) F(E_{th}) \times \int_0^\infty dx x^\gamma e^{-xt}$$

or for large times:

$$A_{LT}(t) \simeq (\text{phase}) \times \Gamma(\gamma+1) P(E_{Th}) F(E_{th}) \times \frac{1}{t^{\gamma+1}}$$

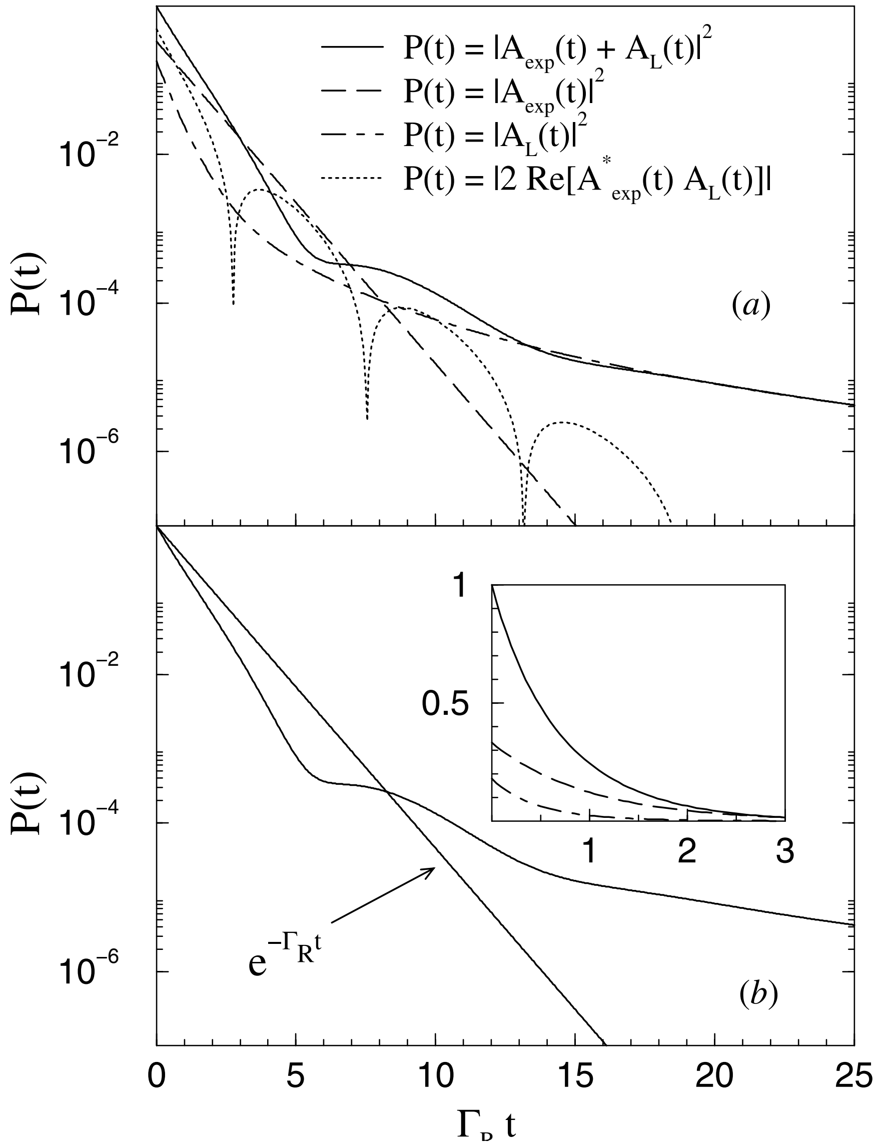
$$A_{LT}(t) = a_{LT} \frac{1}{t^{\gamma+1}}$$

Typical example



$$R = \frac{\Gamma}{m_0 - m_{th}}$$

Extreme broad: sigma meson



Small time also possible by Fock-Krylov method.

- Here we do it straight from definition

$$P(t) = |A(t)|^2 = |\langle\Psi|e^{-iHt}|\psi\rangle|^2$$

It can be calculated at short times to give:

$$P(t) \simeq 1 - (\Delta_\Psi H)t^2$$

connected to

$$\frac{dP(t)}{dt}|_{t=0} = 0 \leftrightarrow \frac{d(e^{-\Gamma t})}{dt}|_{t=0} \neq 0$$

and to Quantum-Zeno effect

$$P_N(t) = \prod_i^N P(t/i) = [1 - (\Delta_\Psi H)(t/N)^2]^N$$

$$\lim_{N\to\infty} P_N(t) = e^0 = 1$$

Quantum mechanically there are three regions with two transitions

small time

$$1 + at^2$$

Seen experimentally

intermediate

$$P(t) = e^{-\Gamma t}$$

Mostly we see this

Large time

$$t^{-(2\gamma+2)}$$

Seen experimentally

Enters again Special Relativity to get the laws in flight

- For the exponential part we can use time dilation
- For the small and large times we can argue that in relativistic mechanics time t should be replaces by proper time s

$$s = t^2 - x^2 = t^2 - v^2 t^2$$

Hence in general we would have while going from rest to flight

$$t \to t/\gamma$$

Crucial steps

$$|\Psi_0(0)\rangle = \int dm\ c(m)\,|m;0\rangle$$

$$|\Psi_p(0)\rangle = U(\Lambda)|\Psi_0(0)\rangle = \int dm\ c(m)\,U(\Lambda)|m;0\rangle$$

$$A_p^U(t) = \big\langle \Psi_p(0) \big| \Psi_p(t) \big\rangle = \big\langle \Psi_p(0) \big| e^{-iHt} \big| \Psi_p(0) \big\rangle$$

$$A_p^U(t) \;= \int dm\, dm'\, c^*(m')\, c(m)\, \langle m', 0 | U^\dagger e^{-iHt} U | m, 0 \rangle$$

$$U^\dagger(\Lambda) e^{-iHt} U(\Lambda) = e^{-iU^\dagger(\Lambda)\, H\, U(\Lambda)}$$

$$U^\dagger(\Lambda)\, P_\mu\, U(\Lambda) = \Lambda_\mu^{\ \nu}\, P_\nu,$$

Relativistic Fock-Krylov as an alternative

The relativistic Fourier transform

$$A_p^U(t) = \int_{m_{th}}^{\infty} dm\, \omega(m)\, e^{-i\sqrt{p^2+m^2}t} \equiv A_p^{FK}(t)$$

With m_0 the resonance mass $p^2 = E_0^2 - m_0^2 = \gamma^2 m_0^2 - m_0^2$ fixed

For narrow resonances the spectral density is very sharp (approx. a Dirac delta); Breit-Wigner,
Therefore we can approximate

$$p^2 + m^2 = \gamma^2 m_0^2 + m^2 - m_0^2 \simeq \gamma^2 m_0^2 + 2m_0(m - m_0)$$

A Taylor expansion : $\sqrt{p^2 + m^2} \simeq m/\gamma + constant\ \ terms$

Ambiguity: the two approaches are not exactly the same

$$P_p(t) \approx P_0(t/\gamma)$$

As we will see later the difference for most of the unstable particles/nuclei is very small which makes it difficult to measure it

Details: some definitions allowing dimensionless quantities

The calculation follows then the path of the Fiock-Krylov method at rest: complex plane and residue theorem (for the exponential part) which phenomenologically is, of course, the most important

$$\mu = \frac{m_{\rm th}}{m_0 - m_{\rm th}},$$

$$P = \frac{p}{m_0 - m_{\rm th}}, \qquad x_0 = \frac{\Gamma_0}{2(m_0 - m_{\rm th})},$$

$$\tau = \Gamma_0 t,$$

$$a(\tau, P) \equiv A_p^{FK}(\tau) = \int_\mu^\infty d\xi\, \lambda(\xi)\, \exp\left(-\frac{i\tau}{2x_0}\sqrt{\xi^2 + P^2}\right)$$

$$\lambda(\xi) \equiv (m_0 - m_{\rm th})\omega((m_0 - m_{\rm th})\xi)$$

Details: the exponential decay law in flight (1)

$$a_e(\tau, P) = -2\pi i \,\mathrm{Res}\left[\lambda(z)\,\exp\left(-\frac{i\tau}{2x_0}\sqrt{z^2+P^2}\right), z = 1+\mu-ix_0\right]$$
$$= R\exp\left[-\frac{i\tau}{2x_0}\sqrt{(\zeta_0+\mu)^2+P^2}\right],$$

$$\zeta_0 \;=\; 1 - ix_0 \qquad\qquad R \;=\; -2\pi i\,\mathrm{Res}\left[\lambda(z)\,, z = \mu+\zeta_0\right]$$

The real part of $\sqrt{(\zeta_0+\mu)^2+P^2}$ we denote by $2x_0\Omega$

The imagineary by $-x_0\sigma$

$$\Omega = \frac{1}{2\sqrt{2}x_0}\left[\sqrt{\left[(P+x_0)^2+(1+\mu)^2\right]\left[(P-x_0)^2+(1+\mu)^2\right]}+P^2+(1+\mu)^2-x_0^2\right]^{1/2},$$
$$\sigma = \frac{1}{\sqrt{2}x_0}\left[\sqrt{\left[(P+x_0)^2+(1+\mu)^2\right]\left[(P-x_0)^2+(1+\mu)^2\right]}-P^2-(1+\mu)^2+x_0^2\right]^{1/2}.$$

Non-relativistic limit

$$\Omega\Big|_{P=0} = \frac{1+\mu}{2x_0},$$
$$\sigma\Big|_{P=0} = 1.$$

Relativistic limit

$$\Omega\Big|_{\substack{P\gg x_0\\ P\gg 1+\mu}} \approx \frac{P}{2x_0},$$
$$\sigma\Big|_{\substack{P\gg x_0\\ P\gg 1+\mu}} \approx \frac{1+\mu}{P}.$$

The important exponential part can be calculated exactly (2)

Important is the deviation from the formula obtained previously by STR

$$P_e(\tau, P) = |a_e(\tau, P)|^2 = |R|^2\, e^{-\sigma\,\tau},$$

General from Fock-Krylov

$$P_e(\tau, 0) = |R|^2\, e^{-\tau},$$

At rest

$$p_e(\tau, \gamma) = |R|^2 \exp\left(-\frac{\tau}{\gamma}\right).$$

From STR

$$P_e^{UR}(\tau, P) = |R|^2 \exp\left(-\frac{\tau}{\gamma}\right)$$

Ultrarelativistic from Fock-Krylov

Exp(Fock-Krylov)/Exp(STR)

$$\frac{P_e(\tau, P)}{p_e(\tau, \gamma)} = \exp\left[-\left(\sigma(P, x_0) - \gamma^{-1}\right)\tau\right].$$

We know already that $\displaystyle\lim_{\gamma\to\infty} \frac{P_e(\tau, P)}{p_e(\tau, \gamma)} = 1.$

Moreover the argument of the exponential $f(\gamma, x_0) = \sigma(P, x_0) - \gamma^{-1}$

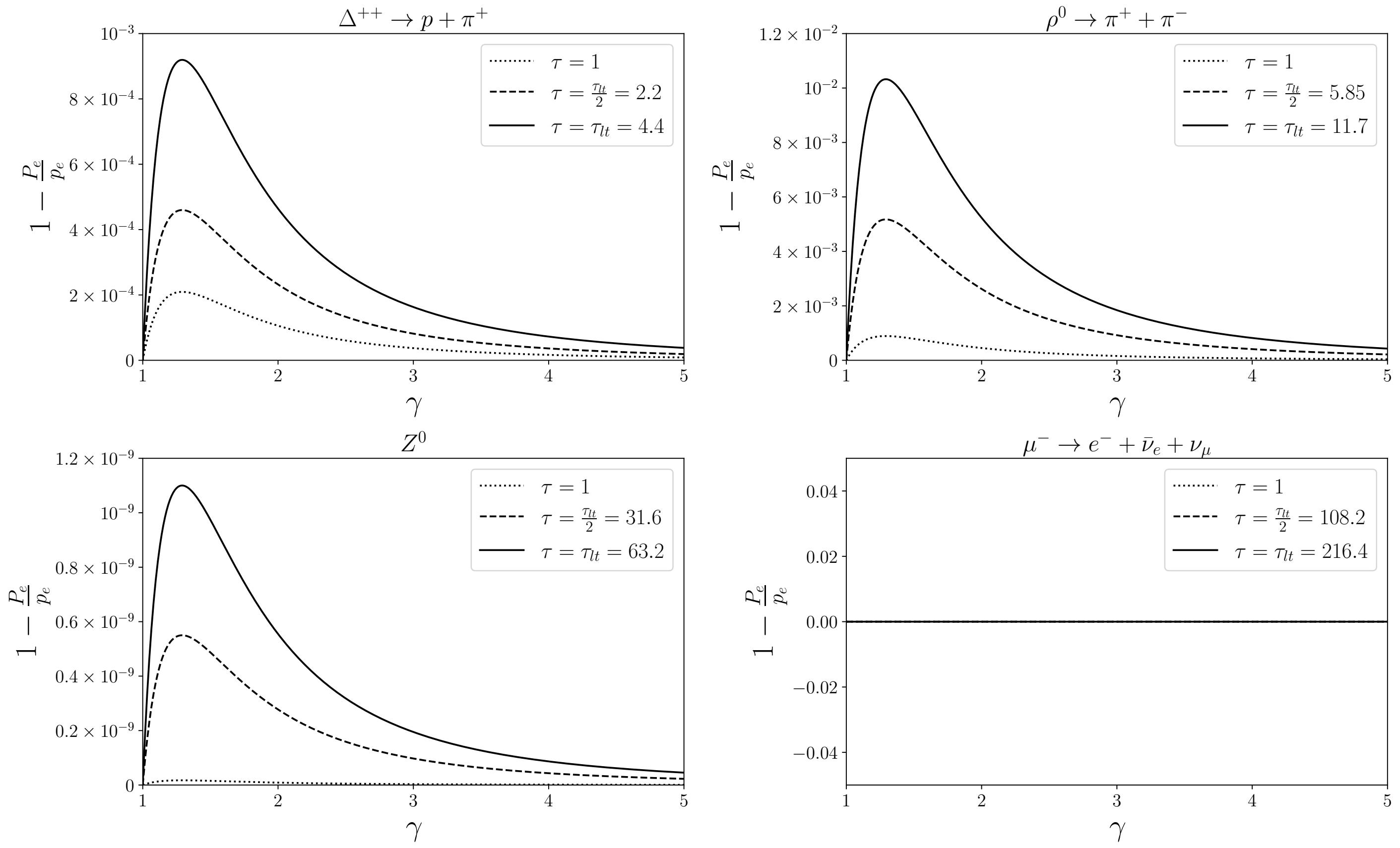
has a maximum at

$$\gamma_c^2 = \frac{5}{3} + \frac{9}{25}\left(\frac{x_0}{1+\mu}\right)^2 - \frac{31}{3125}\left(\frac{x_0}{1+\mu}\right)^4 + \frac{267}{390625}\left(\frac{x_0}{1+\mu}\right)^6 - \frac{573}{9765625}\left(\frac{x_0}{1+\mu}\right)^8 + \frac{33642}{6103515625}\left(\frac{x_0}{1+\mu}\right)^{10} + \cdots.$$

$$\gamma_c^2 \approx \frac{5}{3}$$

$$v = 0.63c$$

Narrow resonances

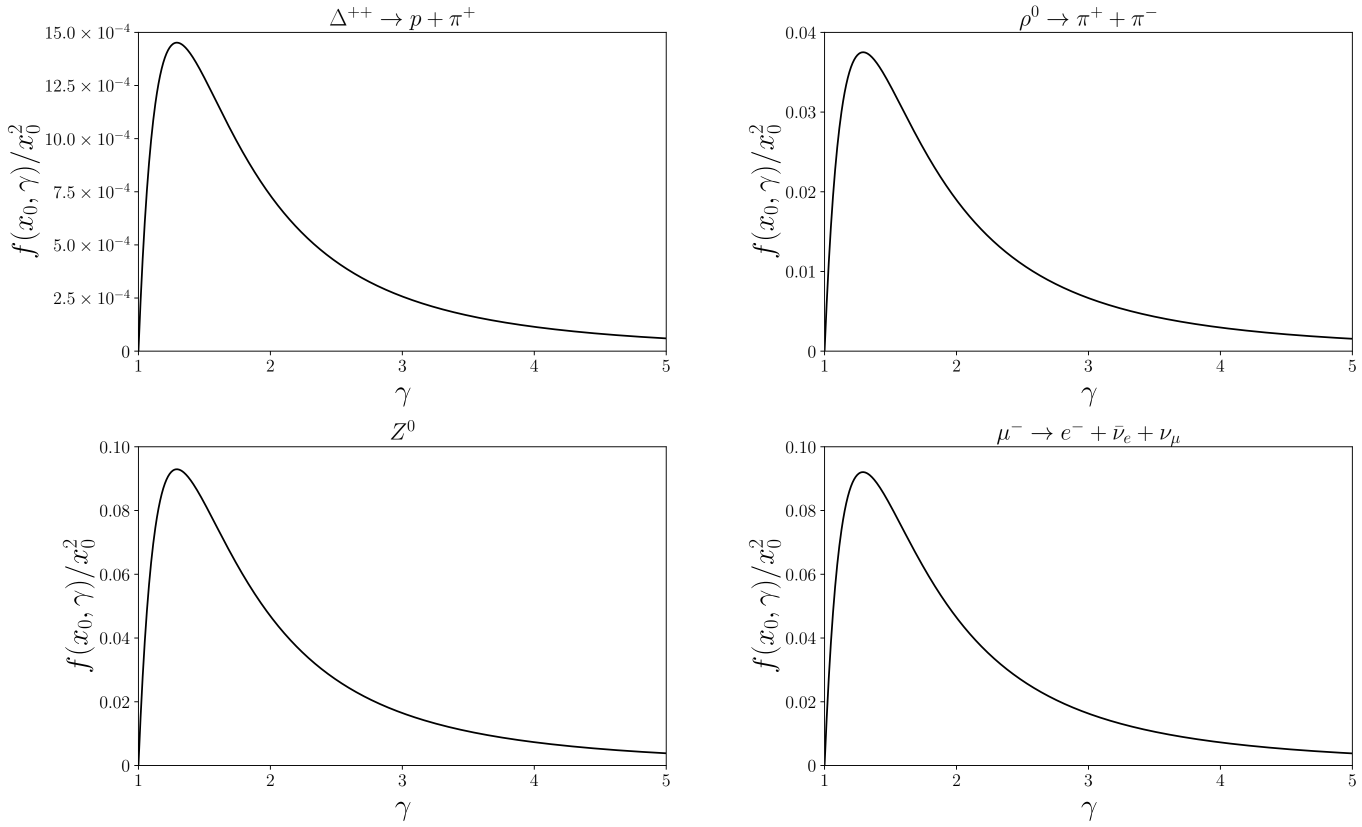


$\Delta^{++} \to p + \pi^+$
$\tau = 1$
$\tau = \frac{\tau_{lt}}{2} = 2.2$
$\tau = \tau_{lt} = 4.4$
$\rho^0 \to \pi^+ + \pi^-$
$\tau = 1$
$\tau = \frac{\tau_{lt}}{2} = 5.85$
$\tau = \tau_{lt} = 11.7$
Z^0
$\tau = 1$
$\tau = \frac{\tau_{lt}}{2} = 31.6$
$\tau = \tau_{lt} = 63.2$
$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$
$\tau = 1$
$\tau = \frac{\tau_{lt}}{2} = 108.2$
$\tau = \tau_{lt} = 216.4$
$1 - \frac{P_e}{p_e}$
γ

Examples

The muon was once the harbinger of relativistic effects in flight

- Plotting the negative argument of the exponential divided by a constant



The effect for the muon

Recall

$$x_0 = \frac{\Gamma_0}{2(m_0 - m_{\text{th}})}; \quad \sim 10^{-18} \qquad f/x_0^2 \sim 10^{-2}$$

$$f \sim 10^{-38} \qquad FK/STR = \frac{P_e(\tau, P)}{p_e(\tau, \gamma)} = e^{-f}$$

Small times in flight
Small times at rest have been seen

Expand the amoplitude

$$a(\tau, P) \equiv A_p^{FK}(\tau) = \int_\mu^\infty d\xi\, \lambda(\xi)\, \exp\left(-\frac{i\tau}{2x_0}\sqrt{\xi^2+P^2}\right)$$

$$a(\tau, P) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i\tau}{2x_0}\right)^n I_n, \quad I_n = \int_\mu^\infty d\xi\, \lambda(\xi)\, \left(\xi^2+P^2\right)^{n/2}.$$

Then the survival probability is

$$S(\tau, P) = |a(\tau, P)|^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\tau}{2x_0}\right)^{2n} \sum_{m=0}^{2n} (-1)^m \binom{2n}{m} I_m I_{2n-m}$$

$$= 1 - \tau^2 \frac{I_2 - I_1^2}{4x_0^2} + \tau^4 \frac{I_4 - 4I_1 I_3 + 3I_2^2}{192 x_0^4} + \cdots .$$

$$\omega(m)$$

is the mass distribution of the resonance (say, Breit-Wigner)

$$\delta_n \equiv \int_{m_{\rm th}}^{\infty} dm\, m^n \omega(m) < \infty,$$

We can consider this quantity as expectation value of m raised to the power n

$$(\tilde{\delta}_2 - \tilde{\delta}_1^2)\, \Gamma_0^2 = \delta_2 - \delta_1^2$$

with

$$\tilde{\delta}_n = \left(\frac{1}{2x_0}\right)^n \int_{\mu}^{\infty} d\xi\, \xi^n \lambda(\xi) = \frac{1}{\Gamma_0^n}\, \delta_n$$

Can be considered as quantum uncertainty: if the particle has a shrp mass, the mass distribution Becomes Dirac delta and the above expression is zero

Small times and uncertainties

Comparison for small times

$$S(\tau, P=0) = 1 - (\tilde{\delta}_2 - \tilde{\delta}_1^2)\tau^2 + O(\tau^4) = 1 - (\delta_2 - \delta_1^2)t^2 + O(t^4)$$

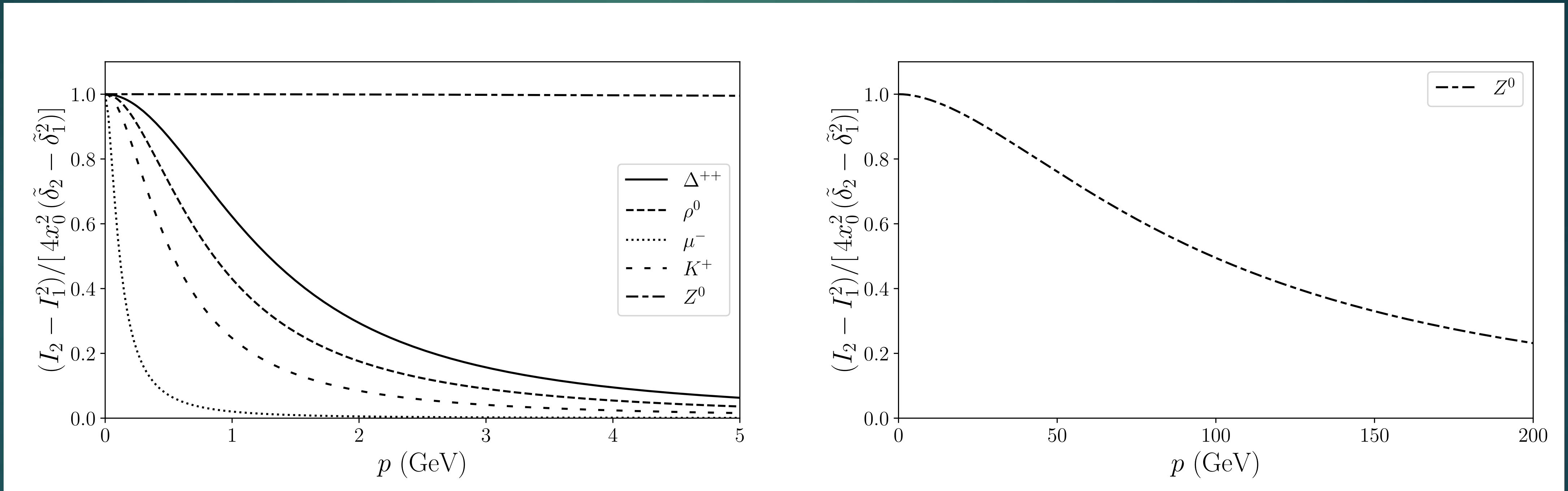
$$S(\tau, P) \; = 1 - \tau^2 \frac{I_2 - I_1^2}{4x_0^2} + \tau^4 \frac{I_4 - 4I_1 I_3 + 3I_2^2}{192 x_0^4} + \cdots$$

$$I_{2s+1} \sim \sum_{v=0}^{\infty} \frac{(-1)^v}{2^{2v}} \binom{2v}{v} \sum_{l=0}^{s+1} \binom{s+1}{l} \frac{1}{P^{2v-2l+1}} \int_{\mu}^{\infty} d\xi \, \lambda(\xi) \, \xi^{2v+2s-2l+2},$$

$$= (2x_0)^{2s+1} \sum_{v=0}^{\infty} (-1)^v \binom{2v}{v} \sum_{l=0}^{s+1} \binom{s+1}{l} \left(\frac{2x_0}{P}\right)^{1-2l} \tilde{\delta}_{2v+2s-2l+2},$$

$$I_{2s} = \sum_{k=0}^{s} \binom{s}{k} P^{2k} \int_{\mu}^{\infty} d\xi \, \lambda(\xi) \, \xi^{2s-2k} = (2x_0)^{2s} \sum_{k=0}^{s} \binom{s}{k} \left(\frac{P}{2x_0}\right)^{2k} \tilde{\delta}_{2s-2k}.$$

Comparisons of the uncertainties



“STR” versus Fock-Krylov

“STR”

$$S(\tau/\gamma, 0) = 1 - (\delta_2 - \delta_1^2)\left(\frac{t^2}{\gamma^2}\right) + \cdots$$

Ultra-relativistic Fock-Krylov

$$S^{UR}(\tau, P) \simeq 1 - \left(\frac{\delta_4 - \delta_2^2}{4m_0^2}\right)\left(\frac{t^2}{\gamma^2}\right)$$

Process	$\delta_2 - \delta_1^2$ (MeV^2)	$(\delta_4 - \delta_2^2)/4m_0^2$ (MeV^2)
Δ^{++}	5682.5	6303.2
ρ^0	26282	42080
Z_0	87245	169040
μ^-	1.2054×10^{-14}	2.3275×10^{-14}
K^+	7.9145×10^{-12}	1.3164×10^{-11}

Full result with a chain of uncertainties

$$
\begin{aligned}
S(\tau,P) = 1 \\
&- \tau^2\Bigg[\left(\tilde{\delta}_4 - \tilde{\delta}_2^2\right)\left(\frac{x_0}{P}\right)^2 + 2\left(\tilde{\delta}_2\tilde{\delta}_4 - \tilde{\delta}_6^2\right)\left(\frac{x_0}{P}\right)^4 - \left(\tilde{\delta}_4^2 + 4\tilde{\delta}_2\tilde{\delta}_6 - 5\tilde{\delta}_8\right)\left(\frac{x_0}{P}\right)^6 \\
&\qquad + 2\left(2\tilde{\delta}_4\tilde{\delta}_6 + 5\tilde{\delta}_2\tilde{\delta}_8 - 7\tilde{\delta}_{10}\right)\left(\frac{x_0}{P}\right)^8 - 2\left(2\tilde{\delta}_6^2 + 5\tilde{\delta}_4\tilde{\delta}_8 + 14\tilde{\delta}_2\tilde{\delta}_{10} - 21\tilde{\delta}_{12}\right)\left(\frac{x_0}{P}\right)^{10} + \cdots\Bigg] \\
&+ \tau^4\Bigg[\frac{3\tilde{\delta}_4^2 - 4\tilde{\delta}_2\tilde{\delta}_6 + \tilde{\delta}_8}{12}\left(\frac{x_0}{P}\right)^4 - \frac{2\tilde{\delta}_4\tilde{\delta}_6 - 3\tilde{\delta}_2\tilde{\delta}_8 + \tilde{\delta}_{10}}{3}\left(\frac{x_0}{P}\right)^6 + \frac{2\tilde{\delta}_6^2 + 9\tilde{\delta}_4\tilde{\delta}_8 - 18\tilde{\delta}_2\tilde{\delta}_{10} + 7\tilde{\delta}_{12}}{6}\left(\frac{x_0}{P}\right)^8 \\
&\qquad - \frac{4\left(\tilde{\delta}_6\tilde{\delta}_8 + 3\tilde{\delta}_4\tilde{\delta}_{10} - 7\tilde{\delta}_2\tilde{\delta}_{12} + 3\tilde{\delta}_{14}\right)}{3}\left(\frac{x_0}{P}\right)^{10} + \cdots\Bigg].
\end{aligned}
$$

$$P_{lt}(\tau) \propto p^{2(\alpha+1)} \frac{1}{t^{2(\alpha+1)}}$$

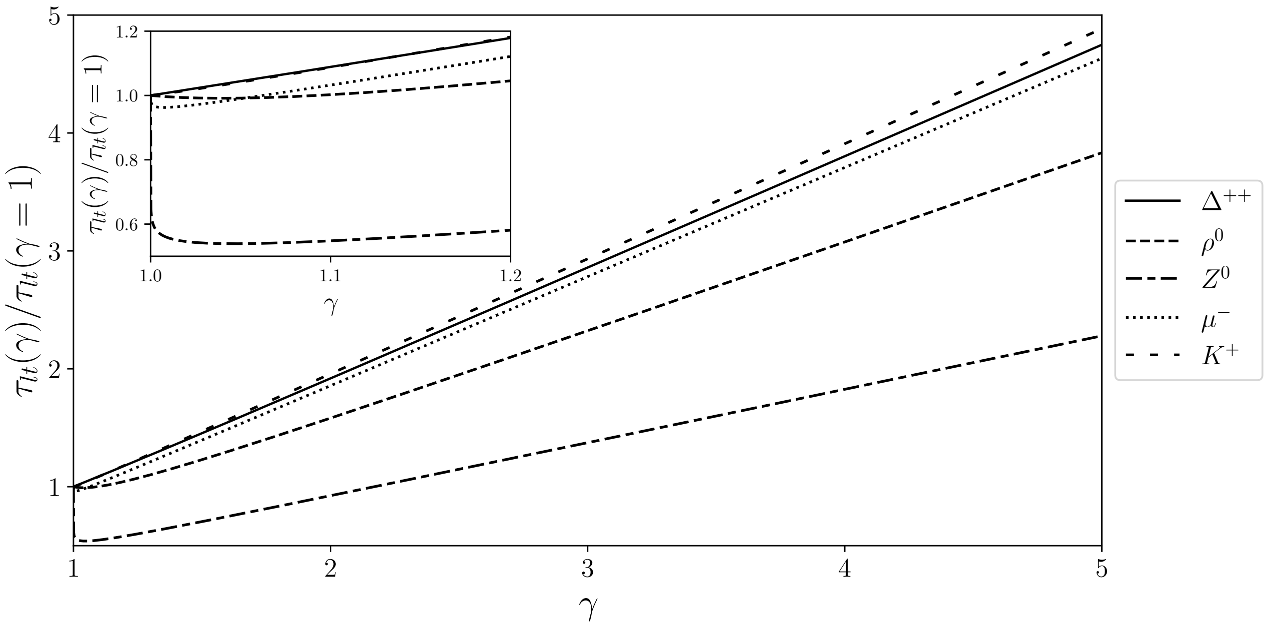
Large time

Transition time: from exponential to power law

		Critical time τ_{lt}				
Process	$w_0 = (\sigma\tau_{lc})\|_{P\to\infty}$	$p = 0$	$p = 0.5$	$p = 1$	$p = 5$	$p = 10$
Δ^{++}	10.193	10.754	11.508	13.537	42.719	83.417
ρ^0	12.458	16.341	16.996	21.549	81.438	161.05
Z^0	30.505	67.065	47.449	45.228	40.096	38.024
μ^-	203.88	220.11	986.43	1940.5	9650.2	19297
K^+	195.75	200.45	279.99	442.92	1992.4	3970.1

Comparison

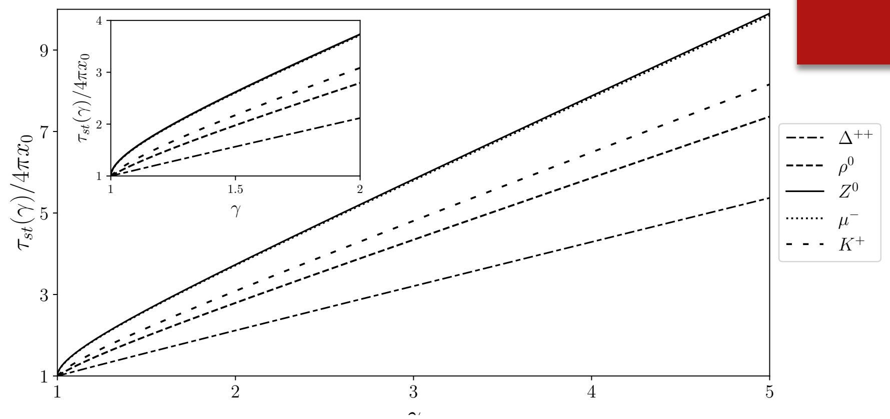
- For some particles there is a critical gamma for which the ratio has a minimum. Best visible for the neutral gauge boson



Transition time: from small times to exponential

- The transition time at rest is

$$4\pi x_0$$



Process	$\tau_{st}/4\pi x_0$				
	$p=0$	$p=0.5$	$p=1$	$p=5$	$p=10$
Δ^{++}	1	1.0914	1.3283	4.4840	8.7991
ρ^0	1	1.4209	2.1962	9.6375	19.182
Z^0	1	1.0055	1.0110	1.0563	1.1157
μ^-	1	9.5228	18.890	94.198	188.38
K^+	1	2.0251	3.5386	16.728	33.391

Collaborators

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History of the subject+referrences + more details

Thank you

Quantum Corrections to the Decay Law in Flight

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