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The derivation of equations for fluctuations and transport in flux-tube geometries

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A self-consistent set of equations for the fast space–time evolution of fluctuations and the slow space–time evolution of density and flows in a toroidal plasma, relevant for simulations using field-aligned coordinates in thin flux tubes, has been derived. The methodology for the derivation of these equations is outlined for a model set of equations for the plasma edge, specific to resistive ballooning modes but readily adaptable to other instabilities. The derivation proceeds by first writing the axisymmetric and fluctuating equations in the usual toroidal coordinate system. These are then transformed to the twisted coordinate flux-tube system. Most simulations which use twisted flux-tube computational grids transform to the field-aligned coordinate system first and then take averages to obtain the slow evolution. They however miss some terms since the two operations, namely, multiscale separation and coordinate transformation, do not necessarily commute, because of subsidiary assumptions on the box size. In the present formulation, all the relevant neoclassical effects such as the Pfirsch–Schlüter current and the Stringer spin-up as well as the toroidal Reynolds stress are properly included. This set of multiscale equations is appropriate for the study of the formation and evolution of transport barriers. [S1070-664X(98)01605-X]

I. INTRODUCTION

The modes that are believed to be responsible for anomalous transport in tokamak plasmas have radial and poloidal correlation lengths a few orders of magnitude smaller than the minor radius. This makes numerical simulations with resolution of all scale-lengths, from these short scale-lengths to the scale-length of the minor radius, impossible to perform with present day computational resources. For this reason, many of the simulations of turbulence are done in a smaller computational volume (narrow flux tubes) constructed around field-aligned coordinates instead of the full torus.[1–6] These numerical grids provide an efficient prescription for addressing the issue of small-scale generated turbulence and transport. However, stemming from this small-scale turbulence is the generation of large-scale flows and modification of the equilibrium parameters. If one attributes the variety of enhanced confined modes observed on tokamaks, like the High (H)[7] mode and the enhanced reversed shear (ERS)[8] modes, to be a result of suppression of fluctuations by self-generated shear flow or steepening of density gradients, a proper system of equations for the fast modes, as well as the large-scale flows, valid for the smaller volume field-aligned geometry have to be used.

The main purpose of this work is to call attention to the incomplete nature of standard simulations, regarding transport. We believe that most simulations (including our own), which resort to the use of the smaller volume field-aligned geometry, do not incorporate all the important terms in the ''slow'' evolution equations, because of an intrinsic flaw in their derivation and the assumptions peculiar to the method. The standard method that is used is to transform a given system of equations to the field-aligned geometry. The ''slow'' evolution of the flow and equilibrium quantities are believed to be contained in the transformed system of equations. The ''slow'' equations can be obtained by an appropriate averaging of these transformed equations. We will show that this sequence of operations, namely, transforming to the twisted coordinate system and averaging, does not yield correct equations. A more natural approach that avoids the above pitfalls is to first derive separate sets of equations for the fast and slow (or axisymmetric part) of the various quantities and then do the transformation to the twisted system. We find that these two operations do not necessarily commute, under the approximations involved. The correctness of our present prescription is evidenced by the fact that our slow equations yield all the relevant neoclassical effects such as the Pfirsch–Schlüter current and the Stringer spin-up,[9] as well as the toroidal Reynolds stress. Our recent study of a quasi three-dimensional model for Low–High (LH) transitions in tokamaks helped us identify the problem.[10] This allowed us to formulate more carefully the general problem in the twisted coordinate system in flux tubes.

The procedure for obtaining the combination of fast and slow equations in toroidal coordinates is similar to that used in Ref. 11, but here the averaging is taken over the toroidal coordinate, i.e., the averaged quantities are axisymmetric. The technique is applied to the problem of resistive ballooning modes in a flux tube with field-aligned geometry. The fast equations agree with those used in previous simulations

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of edge turbulence.[2,6] But the important result is that the slow equations contain information that is not present in the equations used previously in numerical simulations; in particular, the anomalous Stringer spin-up mechanism.[9,10,12]

The methodology we are going to describe could be used on any set of equations that describe a specific model, consisting of transport and turbulence. In our particular case, we apply it to the set of equations that describe a low beta, collisional plasma, obtained from the two-fluid Braginskii equations,[13] which for electrostatic modes consist of the continuity, parallel flow and vorticity equations,

$$\frac{dn}{dt}-2n\kappa\cdot\mathbf{v}_\perp-\nabla\cdot\left(\frac{nc}{\Omega_i B}\frac{d}{dt}\nabla_\perp\phi\right)+\nabla_\parallel(nv_\parallel)=0, \tag{1}$$

$$n\left(\frac{\partial}{\partial t}+\mathbf{v}_\perp\cdot\nabla\right)v_\parallel+nv_\parallel\nabla_\parallel v_\parallel+c_s^2\nabla_\parallel n=0 \tag{2}$$

$$\nabla\cdot\left(\frac{nc}{\Omega_i B}\frac{d}{dt}\nabla_\perp\phi\right)+\frac{B}{e}\nabla_\parallel\left(\frac{\nabla_\parallel\phi}{B\eta_\parallel}-\frac{T_e\nabla_\parallel n}{neB\eta_\parallel}\right)-\frac{2c}{eB}\mathbf{b}\times\kappa\cdot\nabla p=0, \tag{3}$$

where $\mathbf{v}_\perp=(c/B)\mathbf{b}\times\nabla\phi$, and $\kappa=\mathbf{b}\cdot\nabla\mathbf{b}$. For the magnetic field we use the representation $\mathbf{B}=(\hat{\phi}+\Theta\hat{\theta})B_0/R_0(1+\epsilon\cos\theta)$. This is our basic set of equations which contains all the fundamental physics required in our model. The effects of viscosity and temperature variations would modify the results in a quantitative way but do not introduce new effects.

II. METHODOLOGY AND COORDINATES

Since we are interested in turbulence and transport in a tokamak, we assume that all macroscopic quantities (i.e., non-fluctuating) are axisymmetric. Thus, all dependent variables $(n,v_\parallel,\phi)$ are separated into an axisymmetric (macroscopic) component, plus a fluctuating piece related to the micro-instabilities:

$$\xi(x,y,z)=\overline{\xi}(x,y)+\tilde{\xi}(x,y,z), \tag{4}$$

where we use $x=r-a$, $y=a\theta$, and $z=R\varphi$. As a first step, the equations are averaged over the coordinate z to obtain their axisymmetric part $(\overline{\xi})$, and these are then subtracted from the original equations to get the fluctuating, non-axisymmetric components $(\tilde{\xi})$. The axisymmetric equations evolve on a slow time scale, while their fluctuating counterparts are fast evolving.

Once we have the separate sets of equations, these are then transformed to a field-aligned, twisted coordinate system. Notice that in all previous formulations the coordinate transformation is made before any separation in time/space scales is made, so whatever is lost in the approximations made with the transformation applies to fast and slow quantities equally. With this new prescription the transformation and the approximations may affect differently the two scales. In the transformed frame, one coordinate is always aligned with the sheared magnetic field, so its direction depends on x as well as y and z. The transformation is

$$x'=x,\quad y'=y-\frac{\epsilon}{q(x)}z,\quad z'=z+\frac{\epsilon}{q(x)}y. \tag{5}$$

In these coordinates,

$$\nabla_\parallel=\partial/\partial z',$$
$$\frac{\partial}{\partial x}=\frac{\partial}{\partial x'}+\frac{\hat{s}z'}{Rq}\frac{\partial}{\partial y'}, \tag{6}$$

with the magnetic shear $\hat{s}=a\ \ d\ln q/dx$.

In the equations for the axisymmetric quantities, since $\partial/\partial z=0$, all y' variations can be mapped to z' variations by

$$\frac{\partial}{\partial y'}=\frac{q}{\epsilon}\frac{\partial}{\partial z'}, \tag{7}$$

and so they are considered to be only functions of x' and z'. This also implies that the averages over z can be transformed to averages over y', interpreted appropriately.

In order to keep just the physically relevant effects and avoid dealing with complicated equations, we use the small parameter $\epsilon=a/R$, and assume that the scale lengths are ordered as

$$L_n/a\ll 1,\quad L_0/L_n\ll 1, \tag{8}$$

where L_n is the scale length of slow quantities in the radial direction $(L_n^{-1}\sim\partial\ln\overline{\xi}/\partial x)$, $a^{-1}\sim\partial\ln\overline{\xi}/\partial y$, and L_0 represents the correlation length of fluctuations $(L_0^{-1}\sim\partial\ln\tilde{\xi}/\partial x)$. This is applicable to edge plasmas, as previous simulations have shown.[2,6]

III. EQUATIONS

We now apply the method just described to the set of equations (1)–(3). The dependent variables are n, $v_\parallel$, and ϕ. When the equations are averaged over the coordinate z and only the significant terms in ϵ are kept, the resulting set of equations is the following:

$$\frac{\partial\overline{n}}{\partial t}+\frac{c}{B}\left(\frac{\partial\overline{\phi}}{\partial x}\frac{\partial\overline{n}}{\partial y}-\frac{\partial\overline{\phi}}{\partial y}\frac{\partial\overline{n}}{\partial x}\right)-\frac{2c\overline{n}}{BR}\sin\theta\frac{\partial\overline{H}}{\partial x}+\frac{\epsilon}{q}\frac{\partial\overline{nv_\parallel}}{\partial y}+\frac{\epsilon^2}{q^2e\eta_\parallel}\frac{\partial^2\overline{H}}{\partial y^2}+\frac{c}{B}\left\langle\frac{\partial\tilde{\phi}}{\partial x}\frac{\partial\tilde{n}}{\partial y}-\frac{\partial\tilde{\phi}}{\partial y}\frac{\partial\tilde{n}}{\partial x}\right\rangle_z=0, \tag{9}$$

$$\frac{\partial\overline{v_\parallel}}{\partial t}+\frac{c}{B}\left(\frac{\partial\overline{\phi}}{\partial x}\frac{\partial\overline{v_\parallel}}{\partial y}-\frac{\partial\overline{\phi}}{\partial y}\frac{\partial\overline{v_\parallel}}{\partial x}\right)+\frac{\epsilon}{q}\frac{\partial}{\partial y}\frac{\overline{v_\parallel^2}}{2}+c_s^2\frac{\epsilon}{q\overline{n}}\frac{\partial\overline{n}}{\partial y}=0, \tag{10}$$

$$\frac{\partial\overline{\omega}}{\partial t}+\frac{c}{B}\left(\frac{\partial\overline{\phi}}{\partial x}\frac{\partial\overline{\omega}}{\partial y}-\frac{\partial\overline{\phi}}{\partial y}\frac{\partial\overline{\omega}}{\partial x}\right)+\frac{2\Omega_iT}{eR\overline{n}}\sin\theta\frac{\partial\overline{n}}{\partial x}+\frac{\epsilon^2\Omega_iB}{cq^2\overline{n}e\eta_\parallel}\frac{\partial^2\overline{\phi}}{\partial y^2}+\frac{c}{B}\left\langle\frac{\partial\tilde{\phi}}{\partial x}\frac{\partial\tilde{\omega}}{\partial y}-\frac{\partial\tilde{\phi}}{\partial y}\frac{\partial\tilde{\omega}}{\partial x}\right\rangle_z=0, \tag{11}$$

where $H\equiv\phi-\alpha n$ with $\alpha=T/\overline{n}$ held constant (for $T_e\approx T$), the "vorticity,"

$$\omega\equiv\nabla_\perp^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}, \tag{12}$$

and

$$\langle \xi \rangle_z = \frac{1}{2\pi R} \int_0^{2\pi R} \xi dz$$

is the z average. Here, the parallel velocity fluctuations have been neglected, since the resistive ballooning mode is supersonic.[2] Notice that these equations are axisymmetric and hence, two-dimensional. Subtracting these from the original equations, the evolution of the fluctuating quantities is obtained. The fast equations are fully three-dimensional since no symmetry is assumed. Without the parallel flow variations ($\tilde{v}_\parallel \approx 0$), only two equations are needed, namely,

$$\frac{\partial \tilde{n}}{\partial t} + \frac{c}{B}\left\{ \left(\frac{\partial \overline{\phi}}{\partial x}\frac{\partial \tilde{n}}{\partial y} - \frac{\partial \tilde{\phi}}{\partial y}\frac{\partial \overline{n}}{\partial x} \right) + \left[\frac{\partial \tilde{\phi}}{\partial x}\frac{\partial \tilde{n}}{\partial y} - \frac{\partial \tilde{\phi}}{\partial y}\frac{\partial \tilde{n}}{\partial x} \right]_z \right\} - \frac{2c}{BR}\left(\sin\theta \frac{\partial \tilde{H}}{\partial x} + (\cos\theta - \epsilon)\frac{\partial \tilde{H}}{\partial y} \right) + \frac{1}{e\eta_\parallel}\left(\frac{\epsilon^2}{q^2}\frac{\partial^2 \tilde{H}}{\partial y^2} + \frac{\partial^2 \tilde{H}}{\partial z^2} \right) = 0, \tag{13}$$

$$\frac{\partial \tilde{\omega}}{\partial t} + \frac{c}{B}\left\{ \left(\frac{\partial \overline{\phi}}{\partial x}\frac{\partial \tilde{\omega}}{\partial y} - \frac{\partial \tilde{\phi}}{\partial y}\frac{\partial \overline{\omega}}{\partial x} \right) + \left[\frac{\partial \tilde{\phi}}{\partial x}\frac{\partial \tilde{\omega}}{\partial y} - \frac{\partial \tilde{\phi}}{\partial y}\frac{\partial \tilde{\omega}}{\partial x} \right]_z \right\} - \frac{2\Omega_i T}{neR}\left(\sin\theta \frac{\partial \tilde{n}}{\partial x} + (\cos\theta - \epsilon)\frac{\partial \tilde{n}}{\partial y} \right) + \frac{\Omega_i B}{\overline{n} e c \eta_\parallel}\left(\frac{\epsilon^2}{q^2}\frac{\partial^2 \tilde{H}}{\partial y^2} + \frac{\partial^2 \tilde{H}}{\partial z^2} \right) = 0, \tag{14}$$

where we used the simplifying notation $[\tilde{\xi}\tilde{\zeta}]_z \equiv \tilde{\xi}\tilde{\zeta} - \langle \tilde{\xi}\tilde{\zeta} \rangle_z$.

At this point, the transformation to the field-aligned coordinate system given in Eq. (5) is applied. For the axisymmetric quantities, y' variations are linked to z' variations, but the latter is the relevant one. When one considers a domain along a magnetic flux tube, as is done in numerical simulations, the longitudinal variations (along z') are weak compared to those transverse to the tube (y'). Due to this and the fact that turbulence has the same statistical properties all along the z direction, toroidal averages are equivalent to averages over y', defined as

$$\langle \xi \rangle_{y'} = \frac{1}{L_y} \int_0^{L_y} \xi dy'. \tag{15}$$

Here, L_y is the transverse width of the flux tube which is larger than the correlation length L_0. The poloidal coordinate transforms to $\theta = y'/a + z'/Rq$. Furthermore because of the choice of the flux tube box $L_y \ll a$, the y'/a term is neglected in comparison to $z'/Rq \sim 0(1)$. This assumption is well justified for the small-scale fluctuations but not for the axisymmetric ''slow'' variables. Thus, it is for this reason that separation of the ''fast'' fluctuating and ''slow'' average variables be made prior to the transformation to the twisted coordinate system.

We can now write the equations for the five quantities n, $v_\parallel$, ϕ, $\tilde{n}$, and $\tilde{\phi}$, where the overbar is dropped from the averaged variables,

$$\frac{\partial n}{\partial t} + \frac{cq}{B\epsilon}\left(\frac{\partial \phi}{\partial x'}\frac{\partial n}{\partial z'} - \frac{\partial \phi}{\partial z'}\frac{\partial n}{\partial x'} \right) - \frac{2nc}{BR}\sin\left(\frac{z'}{qR}\right)\frac{\partial H}{\partial x'} + \frac{\partial n v_\parallel}{\partial z'} + \frac{1}{e\eta_\parallel}\frac{\partial^2 H}{\partial z'^2} + \frac{c}{B}\left\langle \frac{\partial \tilde{\phi}}{\partial x'}\frac{\partial \tilde{n}}{\partial y'} - \frac{\partial \tilde{\phi}}{\partial y'}\frac{\partial \tilde{n}}{\partial x'} \right\rangle_{y'} = 0, \tag{16}$$

$$\frac{\partial v_\parallel}{\partial t} + \frac{cq}{B\epsilon}\left(\frac{\partial \phi}{\partial x'}\frac{\partial v_\parallel}{\partial z'} - \frac{\partial \phi}{\partial z'}\frac{\partial v_\parallel}{\partial x'} \right) + \frac{\partial}{\partial z'}\frac{v_\parallel^2}{2} + \frac{c_s^2}{n}\frac{\partial n}{\partial z'} = 0, \tag{17}$$

$$\frac{\partial \omega}{\partial t} + \frac{cq}{B\epsilon}\left(\frac{\partial \phi}{\partial x'}\frac{\partial \omega}{\partial z'} - \frac{\partial \phi}{\partial z'}\frac{\partial \omega}{\partial x'} \right) + \frac{2\Omega_i T}{enR}\sin\left(\frac{z'}{qR}\right)\frac{\partial n}{\partial x'} + \frac{\Omega_i B}{cne\eta_\parallel}\frac{\partial^2 H}{\partial z'^2} + \frac{c}{B}\left\langle \frac{\partial \tilde{\phi}}{\partial x'}\frac{\partial \tilde{\omega}}{\partial y'} - \frac{\partial \tilde{\phi}}{\partial y'}\frac{\partial \tilde{\omega}}{\partial x'} \right\rangle_{y'} = 0, \tag{18}$$

$$\frac{\partial \tilde{n}}{\partial t} + \frac{c}{B}\left\{ \left(\frac{\partial \phi}{\partial x'}\frac{\partial \tilde{n}}{\partial y'} - \frac{\partial \tilde{\phi}}{\partial y'}\frac{\partial n}{\partial x'} \right) + \left[\frac{\partial \tilde{\phi}}{\partial x'}\frac{\partial \tilde{n}}{\partial y'} - \frac{\partial \tilde{\phi}}{\partial y'}\frac{\partial \tilde{n}}{\partial x'} \right]_{y'} \right\} - \frac{2cn}{BR}C(\tilde{H}) + \frac{1}{e\eta_\parallel}\frac{\partial^2 \tilde{H}}{\partial z'^2} = 0, \tag{19}$$

$$\frac{\partial \tilde{\omega}}{\partial t} + \frac{c}{B}\left\{ \left(\frac{\partial \phi}{\partial x'}\frac{\partial \tilde{\omega}}{\partial y'} - \frac{\partial \tilde{\phi}}{\partial y'}\frac{\partial \omega}{\partial x'} \right) + \left[\frac{\partial \tilde{\phi}}{\partial x'}\frac{\partial \tilde{\omega}}{\partial y'} - \frac{\partial \tilde{\phi}}{\partial y'}\frac{\partial \tilde{\omega}}{\partial x'} \right]_{y'} \right\} - \frac{2\Omega_i T}{neR}C(\tilde{n}) + \frac{\Omega_i B}{nec\eta_\parallel}\frac{\partial^2 \tilde{H}}{\partial z'^2} = 0, \tag{20}$$

where the vorticities are now given by

$$\omega = \frac{\partial^2 \phi}{\partial x'^2}, \tag{21}$$

$$\tilde{\omega} = \left(\frac{\partial \tilde{\phi}}{\partial x'} + \frac{\hat{s}z'}{Rq}\frac{\partial \tilde{\phi}}{\partial y'} \right)^2 + \frac{\partial^2 \tilde{\phi}}{\partial y'^2}. \tag{22}$$

Notice that the presence of shear is reflected only here and in the curvature operator defined by,

$$C(\tilde{n}) = \left(\sin\left(\frac{z'}{qR}\right)\frac{\partial \tilde{n}}{\partial x'} + \left[\cos\left(\frac{z'}{qR}\right) - \epsilon + \frac{\hat{s}z'}{qR}\sin\left(\frac{z'}{qR}\right) \right]\frac{\partial \tilde{n}}{\partial y'} \right). \tag{23}$$

The evolution of the slow quantities is independent of the magnetic shear.

The coupling of fast and slow equations given in this way describes clearly the influence one type of quantities has on the other. The turbulence gives rise to transport (through the terms $\langle \ldots \rangle_{y'}$), and the modification of the macroscopic profiles due to transport affects the level of turbulence. Obviously, the slow and fast equations evolve on two different time-scales and the averages of the fast fluctuations that enter the slow equations are slowly varying. On the other hand the effect of the slow transport on the fast fluctuations is through the long time modification of equilibrium parameters such as the radial density profile and the generation of sheared electric fields. The slow equations (16)–(18) contain all the expected transport properties, including neoclassical and turbulent transport as discussed in the next section.

IV. PLASMA SPIN-UP

We will show here that our multiscale equations are capable of describing two important properties of toroidal plasmas: Pfirsch–Schlüter (PS) transport and Stringer spin-up.[9] The latter is not present in the equations used for numerical simulations of turbulence, as will become apparent below. The Pfirsch–Schlüter return current may be obtained from Eq. (18) in steady state without the fluctuations contribution, thus balancing the last two terms, corresponding to the curvature and parallel current. Two integrations give the PS potential (neglecting diamagnetic contributions),

$$\phi_0=\frac{2c}{B}\,\eta_\parallel TRq^2\,\sin\left(\frac{z'}{Rq}\right)\frac{dn}{dx'}. \tag{24}$$

Of course, when this is substituted in the third term of Eq. (16), the well-known Pfirsch–Schlüter diffusion results.

We now derive the anomalous Stringer spin-up. To keep the discussion simple we will do this in the subsonic limit, i.e., we will assume that $\partial/\partial t \ll c_s/qR$ for the averaged quantities. We will also ignore the diamagnetic terms. We expand $\phi=\phi_0+\phi_1$. Thus, from Eq. (18), to lowest order this term would give $\partial^2\phi_0/\partial z'^2=0$, or $\phi_0=\phi_0(x')$. The next order contribution given by the other terms can be obtained by integrating the equation over z', assuming periodicity in z',[15] thereby annihilating the large term. In doing so, the convective terms cancel since they involve z' derivatives of ϕ_0 and the result is

$$\frac{\partial\omega_0}{\partial t}+\frac{2Bc_s^2}{cnR}\oint \sin\left(\frac{z'}{qR}\right)\frac{\partial n}{\partial x'}\frac{dz'}{2\pi qR}=0, \tag{25}$$

where for the moment we have ignored the Reynold's stress term. To determine n we now consider Eq. (17). To lowest order the dominant term is the pressure-gradient term, and therefore $c_s^2\partial n_0/\partial z'=0$, implying $n_0=n_0(x')$. In Eq. (25) we need the z'-dependent component of n as function of ϕ_0, so that the integral does not vanish. This is obtained from Eq. (17) to next order and Eq. (16) in the lowest order. Taking the average over z' of Eq. (16) we can express the time derivative of n_0 in terms of the z'-average of the fluctuation term. Thus, using the notation introduced after Eq. (14), we obtain,

$$-\frac{2c}{BR}\sin\left(\frac{z'}{qR}\right)\frac{\partial\phi_0}{\partial x'}+\frac{\partial v_\parallel}{\partial z'}+\frac{c}{Bn_0}\left[\left\langle\frac{\partial\tilde{\phi}}{\partial x'}\frac{\partial\tilde{n}}{\partial y'}-\frac{\partial\tilde{\phi}}{\partial y'}\frac{\partial\tilde{n}}{\partial x'}\right\rangle_{y'}\right]_{z'}=0, \tag{26}$$

$$\frac{\partial v_\parallel}{\partial t}+\frac{cq}{B\epsilon}\frac{\partial\phi_0}{\partial x'}\frac{\partial v_\parallel}{\partial z'}+\frac{c_s^2}{n_0}\frac{\partial n_1}{\partial z'}=0. \tag{27}$$

Integrating Eq. (26) to obtain $v_\parallel$, substituting it in Eq. (27) and integrating, one can find $n_1(x',z')$. This is in turn substituted in Eq. (25), which gives a single equation for the vorticity. It is actually more convenient to integrate Eq. (25) over x' to have an equation for $d\phi_0/dx'$, which is proportional to the zeroth-order $E\times B$-drift. We get

$$(1+2q^2)\frac{\partial}{\partial t}\left(\frac{d\phi_0}{dx'}\right)=\left(\frac{d\phi_0}{dx'}\right)\frac{2cq}{an_0B}\oint\cos\left(\frac{z'}{qR}\right)\times\left\langle\frac{\partial\tilde{\phi}}{\partial y'}\frac{\partial\tilde{n}}{\partial x'}-\frac{\partial\tilde{\phi}}{\partial x'}\frac{\partial\tilde{n}}{\partial y'}\right\rangle_{y'}\frac{dz'}{2\pi}, \tag{28}$$

where a term associated with the time derivative of the turbulent transport has been omitted, assuming it to be small. Note that we recover the PS enhancement in the factor $2q^2$. It is seen that the asymmetry in the fluctuation-driven radial transport may produce a spin-up instability $[(\partial/\partial t)(\partial\phi_0/\partial x')>0]$, provided the turbulent flow Γ, defined through

$$\frac{1}{x'}\frac{\partial}{\partial x'}x'\Gamma=-\frac{c}{B}\left\langle\frac{\partial\tilde{\phi}}{\partial y'}\frac{\partial\tilde{n}}{\partial x'}-\frac{\partial\tilde{\phi}}{\partial x'}\frac{\partial\tilde{n}}{\partial y'}\right\rangle_{y'}, \tag{29}$$

is peaked at the outboard side of the torus.[14]

When all the fluctuation terms in Eqs. (16) and (17) are kept, Eq. (28) takes the form

$$\begin{aligned}(1+2q^2)\frac{\partial}{\partial t}\frac{d\phi_0}{dx'}
&=-\frac{c}{B}\oint\frac{dz'}{2\pi qR}\int dx'\left\langle\frac{\partial\tilde{\phi}}{\partial x'}\frac{\partial\tilde{\omega}}{\partial y'}-\frac{\partial\tilde{\phi}}{\partial y'}\frac{\partial\tilde{\omega}}{\partial x'}\right\rangle_{y'}\\
&\quad-\frac{1}{R}\oint\frac{dz'}{\pi}\cos\left(\frac{z'}{qR}\right)\left\langle\frac{\partial\tilde{\phi}}{\partial x'}\frac{\partial\tilde{v}_\parallel}{\partial y'}-\frac{\partial\tilde{\phi}}{\partial y'}\frac{\partial\tilde{v}_\parallel}{\partial x'}\right\rangle_{y'}\\
&\quad+\frac{d\phi_0}{dx'}\frac{2c}{anBR}\int\cos\left(\frac{z'}{qR}\right)\\
&\quad\times\left\langle\frac{\partial\tilde{\phi}}{\partial y'}\frac{\partial\tilde{n}}{\partial x'}-\frac{\partial\tilde{\phi}}{\partial x'}\frac{\partial\tilde{n}}{\partial y'}\right\rangle_{y'}\frac{dz'}{2\pi qR}.\end{aligned} \tag{30}$$

The first two terms are the Reynolds stresses, which can also make the plasma rotate.

Now, we can apply the same analysis to the set of equations that are used in the numerical simulations of turbulence, to show that the spin-up is not obtained, and therefore they do not provide an adequate description of the transport. We start from the equations in twisted coordinates for resistive ballooning modes of Refs. 2, 6. These are three-dimensional equations unlike the two-dimensional slow equations used above. The system of equations is more difficult to reduce due to the two scales involved. The main difference from our equations for the slow component is that the convective terms involve derivatives with respect to the transverse, short-scale coordinate y' instead of the field-aligned z', and since there is no distinction between fluctuations and averaged quantities, they contain the fluctuation-induced transport as well as large-scale convection. If the simulations were done in the full torus, there would be no problem. This is because for the full torus case, no specific ordering would be imposed on the quantities involved, and, for the axisymmetric perturbations, Eq. (7) would be valid. However in the flux tube geometry, the box size in y' is limited to sizes much smaller than than the minor radius so that contributions of the axisymmetric perturbations to the **E×B** convection term are not included, as discussed in more detail below. To demonstrate the consequences of these approximations, once again we can write each of the density and the potential as consisting of an average piece and a fluctuating piece denoted with a tilde ($\widetilde{\ }$). The average in the twisted coordinate system is an average with respect to y' as defined by Eq. (15). This implies that $\langle\phi\rangle_{y'} = \langle\phi\rangle_{y'}(x',z')$. Then, we may take the y'-average of the equations to obtain their evolution in the transport scale, which reduces the analysis to two dimensions, as before.

We can start again by integrating the vorticity equation over a cycle in z' to eliminate the parallel current term. We can also integrate it in x' to have an equation for the $E\times B$ drift instead of the vorticity [here ω is defined by Eq. (22)]. The resulting dimensionless equation for the average in normalized variables is

$$\oint \left(\frac{\partial}{\partial t} \frac{\partial\langle\phi\rangle_{y'}}{\partial x'} + \int dx' \langle\{\phi,\omega\}\rangle_{y'} + \sin z' \langle n\rangle_{y'} \right) \frac{dz'}{2\pi} = 0, \tag{31}$$

where we introduced the Poisson bracket,

$$\{A,C\} = \frac{\partial A}{\partial x'}\frac{\partial B}{\partial y'} - \frac{\partial A}{\partial y'}\frac{\partial B}{\partial x'}. \tag{32}$$

Next, within the subsonic regime ($\gamma = c_s t_0/qR > 1$), the parallel flow equation to lowest order yields $\langle n_0\rangle_{y'} = \langle n_0\rangle_{y'}(x')$. The next order equation for $\langle v_\parallel\rangle_{y'}$ gives the z'-dependent density $\langle n_1\rangle_{y'}(x',z')$ that is needed in Eq. (31). This is taken in combination with the continuity equation for the average part to lowest order. As before, the time derivative of the density is eliminated by taking the z'-average. Thus we get the following normalized equations, analogous to Eqs. (26)–(27) (neglecting the diamagnetic term):

$$\langle\{[\tilde\phi]_{z'}, \tilde n_0\}\rangle_{y'} - \delta\left[\sin z' \frac{\partial\langle\phi\rangle_{y'}}{\partial x'}\right]_{z'} + \gamma\frac{\partial\langle v_\parallel\rangle_{y'}}{\partial z'} = 0, \tag{33}$$

$$\frac{\partial\langle v_\parallel\rangle_{y'}}{\partial t} + \langle\{\tilde\phi, \tilde v_\parallel\}\rangle_{y'} + \gamma\frac{\partial\langle n_1\rangle_{y'}}{\partial z'} = 0, \tag{34}$$

where $\delta = 2L_n/R$. The first thing that we notice is that Eqs. (26) and (33) are basically the same. The fundamental difference is between Eqs. (27) and (34). In Eq. (34) if we were to assume that the fluctuating parallel flow is negligible, as we have done earlier, then the y' average of the Poisson bracket term vanishes. However Eq. (27) has the extra term (the second term on the left hand side) which is the essential one for driving the Stringer flow. This drive term comes from the **E×B** convection term in the parallel momentum equation $[\mathbf{v}_E \cdot \nabla v_\parallel = (\partial\phi_0/\partial x')(\partial v_\parallel/\partial z')$ in Eq. (27)]. It arises from the axisymmetric $m=1$ component (proportional to $\sin\theta$) of the parallel flow interacting with an axisymmetric $m=0$ component of the electrostatic potential, leading to v_{E0}. In the standard flux-tube approach, the assumption that $k_\perp \equiv \partial/\partial y' \gg k_\parallel \equiv \partial/\partial z'$ throws away this **E×B** convection term for all axisymmetric modes. However, the assumption $k_\perp \gg k_\parallel$ is not quite valid for the axisymmetric $m=1$ component, which has $k_\perp \sim 1/a$ and $k_\parallel \sim 1/qR$.

An equation for $\langle\phi\rangle_{y'}$ can be derived as before, from Eqs. (31), (33), (34), for the case considered here when $\tilde v_\parallel = 0$, obtaining

$$\frac{\partial}{\partial t} \oint \frac{dz'}{2\pi}(1+4q^2\sin^2 z')\frac{\partial\langle\phi\rangle_{y'}}{\partial x'} = -\oint \frac{dz'}{2\pi}\frac{\cos z'}{\gamma^2}\left\langle \frac{\partial}{\partial t}\left\{\int [\tilde\phi]_{z'} dz, n_0\right\}\right\rangle_{y'} - \oint \frac{dz'}{2\pi}\int dx' \langle\{\phi,\omega\}\rangle_{y'}. \tag{35}$$

As one can readily see, there are no terms on the right hand side proportional to the averaged drift velocity $(d\langle\phi\rangle_{y'}/dx')$, that could give a spin-up. All terms involve the fluctuating component of ϕ. The second term is the usual Reynolds stress which can drive rotation. On the left hand side, we can assume $\langle\phi\rangle_{y'}$ to be independent of z', since it is a slowly varying quantity, recovering the PS enhancement factor. As it turns out, all possible contributions from the slowly varying quantities have been filtered by the ordering assumed in the flux-tube geometry. Therefore, the anomalous Stringer spin-up mechanism cannot be obtained, although the neoclassical PS contribution $2q^2$ does appear properly.

V. DISCUSSION AND CONCLUSIONS

We have shown that the slow equations include the Stringer spin-up which, in addition to the Reynolds stress that is also included, can give rise to sheared radial electric fields, which are associated with the observed poloidal rotation in tokamaks during H mode operation. The fast equations describe the evolution of the resistive ballooning modes, including the modifications introduced by the transport to the axisymmetric quantities. As such, the model is self-consistent. Therefore, the effect of fluctuation quenching due to a sheared radial electric field is present. We have kept in the continuity equation the relatively small contribution from the polarization drift, which is responsible for the density contribution in the function H, and the resistive term. Although these are formally small and might be dropped, we include them in order to be able to compare our equations with those used in the previous numerical studies of turbulence.[2,6] We have already shown that the latter equations do not yield the Stringer spin-up, but here we make more explicit the difference between the two sets of equations. Qualitatively, the reason the spin-up is not present, is that the flux tube approach is restricted to scales of the order of the tube width and cannot account for scale-lengths typical of the axisymmetric quantities, of the order of the minor radius. When the usual procedure of averaging is applied to the full equations to get the transport, some kind of filtering takes place. This can be easily seen by directly averaging the flux-tube equations used in the simulations to obtain separate equations for the averaged parts and the fluctuations, as we did here. As we already mentioned, the average in twisted geometry has to be made over the y' coordinate which is the fast varying one, and we write,

$$\phi(x',y',z')=\langle\phi\rangle_{y'}(x',z')+\tilde{\phi}(x',y',z'), \tag{36}$$

and the same for n and $v_\parallel$. This is used in the normalized equations of Refs. 2, 6, and we take the average over y' to obtain the slow component, and subtract it from the original equation to get the fast equation. For the normalized vorticity, for instance, the following pair of equations results, when periodic boundary conditions in y' are assumed:

$$\frac{\partial\langle\omega\rangle_{y'}}{\partial t}+\sin\left(\frac{z'}{qR}\right)\frac{\partial\langle n\rangle_{y'}}{\partial x'}+\frac{\partial^2\langle H\rangle_{y'}}{\partial z'^2}+\left\langle\frac{\partial\tilde{\phi}}{\partial x'}\frac{\partial\tilde{\omega}}{\partial y'}-\frac{\partial\tilde{\phi}}{\partial y'}\frac{\partial\tilde{\omega}}{\partial x'}\right\rangle_{y'}=0, \tag{37}$$

$$\frac{\partial\tilde{\omega}}{\partial t}+\frac{\partial\langle\phi\rangle_{y'}}{\partial x'}\frac{\partial\tilde{\omega}}{\partial y'}-\frac{\partial\tilde{\phi}}{\partial y'}\frac{\partial\langle\omega\rangle_{y'}}{\partial x'}+\left[\frac{\partial\tilde{\phi}}{\partial x'}\frac{\partial\tilde{\omega}}{\partial y'}-\frac{\partial\tilde{\phi}}{\partial y'}\frac{\partial\tilde{\omega}}{\partial x'}\right]_{y'}-C(\tilde{n})+\frac{\partial^2\tilde{H}}{\partial z'^2}=0. \tag{38}$$

The first thing to notice is that Eq. (37) does not have the convective term that appears in Eq. (18). On the other hand Eq. (38) for the fluctuations does agree with Eq. (20) for periodic boundaries. It is then clear that the $n=0$ component is the one that is not accounted for properly in the flux-tube equations, when slow and fast contributions are not considered separately. The missing terms in the slow equations are those responsible for the convection along the field lines (involving $\partial/\partial z'$), and without them it is not possible to have the Stringer spin-up, which is due to the field-aligned return flows.[9,12]

It is noteworthy that the structure of the equations (16)–(20) is the same as the one used in Ref. 10, where only one unstable mode was retained. The main difference from the single-mode analysis is that now there are averages of the fluctuations over the transversal coordinate. This analogy validates the results found in Ref. 10 for the L–H transition, since they are now based on a more complete model.

The more general case of variable temperature and viscosity has been worked out and includes the important effect of magnetic pumping, which opposes the spin up. We do not present this work here to make the presentation as clear as possible. The extra terms just produce lower growth rates and rotation speeds, but the structure of the equations is equivalent.

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