Plasma equilibria for the case of a strong coupling of parallel and E×B flows

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It is shown that the plasma of a tokamak can adopt equilibria with large pressure and electric potential variations along a magnetic flux surface, due to the strong coupling between parallel and perpendicular flows. Different types of solutions to the equilibrium two-fluid equations are found that describe actual plasma states. In particular, one of the equilibria is such that some plasma parameters, like the plasma pressure, can have maximum values in the vicinity of the X point of the separatrix, in agreement with the recent observations in DIII-D [J. L. Luxon and L. G. Davis, Fusion Technol. **8**, 441 (1985)]. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2217931]

I. INTRODUCTION

Traditionally, the plasma equilibrium solutions in a tokamak have been assumed to be those derived from the ideal magnetohydrodynamic (MHD) equations which, in particular, require that the pressure and the poloidal current are flux surface quantities: $p=p(\psi)$ and $I=I(\psi)$. This is the lowest order equilibrium state usually used for obtaining global magnetic configurations, which can be the starting point for more refined solutions and may also be used for stability studies of MHD modes. When no plasma flows are present, the equilibrium is described by the Grad-Shafranov equation, but in presence of a toroidal flow this is modified to the Maschke-Perrin equation.[1] More general equilibria including other classes of flows have been studied from the ideal Magnetohydrodynamic model in Refs. 2 and 3. In those cases it is seen that the plasma pressure ceases to be a flux surface quantity, becoming, for instance, $p=p(\psi,R)$, R being the coordinate of the torus mayor radius, and thus the constant pressure surfaces are shifted with respect to the magnetic flux surfaces. A multifluid analysis has also been done, to extend the MHD results,[4] where the formal solutions are obtained in terms of a generalized flux function, Ψ_α for each species, that includes the vorticity of the corresponding flow, in addition to the usual magnetic flux function ψ, and it turns out that the pressure is not a function of Ψ_α. All these descriptions are meant to compute the global equilibrium of the toroidal plasma and therefore they focus on special types of flows that are more commonly found in tokamaks such as toroidal or poloidal rotations, but they show that in these particular cases, the presence of flows causes the pressure not to be constant along magnetic field lines, as it is in static equilibria.

On the experimental side, there is also evidence of the variation of the pressure over flux surfaces. One such observation is the one made in DIII-D,[5] that the electric potential and the electron pressure have maxima near the X point in low confinement (L-mode) plasmas.[6] This drives an $\mathbf{E}\times\mathbf{B}$ circulation about the X point that takes plasma across flux surfaces and the separatrix. The observations were also supported by numerical simulations with the code UEDGE[7] that self-consistently included $\mathbf{E}\times\mathbf{B}$ and ∇B drifts. Although the first interpretation given in Ref. 6 claimed that the pressure and the electron temperature were constant on a magnetic surface, and it was the ion temperature the one that varied along the magnetic field direction, there is not good evidence to support the claim, and we argue here that the results can be more easily explained in terms of a variable pressure over flux surfaces.

Another example may be the establishment of the so-called multifaceted asymmetric radiation from the edge (MARFE), at the inner edge of tokamaks, where the increase of plasma density accompanied by a reduction of the plasma temperature seems not to cause a reduction of plasma density at the outer side of the torus,[8] as it should be for the case of the radiation-condensation instability. Thus, a quasiequilibrium state with $p\neq p(\psi)$ arises. These examples show in a more local sense that there is a variation of the pressure along field lines, probably associated with plasma flows.

The importance of the flow direction to produce field-aligned pressure variations was pointed out in Ref. 9, where it was demonstrated that a strong coupling of $\mathbf{E}\times\mathbf{B}$ and parallel flows can result in a large pressure change along the magnetic flux surfaces. For this effect to be of relevance it is necessary to have a shear in the $\mathbf{E}\times\mathbf{B}$ flow that breaks the symmetry in the direction perpendicular to $\mathbf{B}$. This effect can be important for many physical applications including physics of plasma transport in tokamak divertor and X-point regions.

Other situation where the analysis of a variable pressure over flux surfaces is of relevance is the startup phase of a tokamak, when the magnetic field is mainly toroidal, as the plasma current has not been yet built up. This was studied in Ref. 10 where it was shown that the $\mathbf{E}\times\mathbf{B}$ drift becomes very important in determining the radial spreading of the plasma, when the ratio of the poloidal to the toroidal field (B_p/B_T) is very small. These results also suggest that the $\mathbf{E}$

$\times\mathbf{B}$ drift effects can be important for the region near the X point, where the ratio B_p/B_T is small also.

In this article we focus on an analytic study of the impact of a strong coupling of parallel and perpendicular flows on plasma equilibrium. In Sec. II we formulate the problem and describe the kind of geometries where our analysis is applicable. In Sec. III we obtain some solutions and discuss their meaning and the relevance to specific cases, in particular, the role it can play to explain the experimental results from DIII-D. Finally, in Sec. IV we discuss the relevance of our results and give the conclusions, pointing out to further studies in this regard.

II. BASIC PLASMA EQUATIONS

We will consider a plasma of electrons and ions with the same temperature. We adopt a cylindrical approximation for a tokamak with a strong "toroidal" magnetic field in the z direction, B_z=const., and assume that there is no z dependence of the plasma parameters. Then the velocity of the flow, $\mathbf{v}$, can be written as follows:

$$\mathbf{v} = \mathbf{b}v_{\parallel} + \mathbf{v}_{\perp}, \tag{1}$$

where $\mathbf{b}\equiv\mathbf{B}/B$ is the unit vector along the magnetic field $\mathbf{B}$; $v_{\parallel}$ is the parallel velocity, and $\mathbf{v}_{\perp}=V_E(\mathbf{b}\times\nabla\varphi)(B_z/B)$ is the perpendicular fluid velocity caused by the $\mathbf{E}\times\mathbf{B}$ drift, φ is the electrostatic potential multiplied by electron charge and V_E is a normalization constant.

Taking into account that $|B_{\perp}|/B_z\sim\beta\ll 1$ we have

$$\mathbf{b} \cong \mathbf{e}_z + \beta(\mathbf{e}_z\times\nabla\psi), \tag{2}$$

where $\psi\equiv\psi(x,y)$ is the magnetic flux function, and

$$\mathbf{v}_{\perp} \cong V_E(\mathbf{b}\times\nabla\varphi). \tag{3}$$

Then, considering the cold ion approximation and no parallel current we have the equations of continuity

$$\nabla\cdot(n\mathbf{v}) = 0, \tag{4}$$

(n is the plasma density), parallel plasma momentum balance

$$\{(n\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla P/\mathrm{M}\}\cdot\mathbf{b} = 0, \tag{5}$$

($P=nT$ is the plasma pressure, T is the electron temperature, and M is the ion mass), and the parallel electron balance

$$\{-\nabla\varphi + \alpha\nabla T + \nabla P/n\}\cdot\mathbf{b} = 0, \tag{6}$$

where α is the thermal force coefficient.

Recalling that $\partial(\cdots)/\partial z=0$, the continuity equation (4) can be written as follows:

$$n\mathbf{v} = \mathbf{e}_z n v_z + \mathbf{e}_z\times\nabla G, \tag{7}$$

where v_z is the z component of plasma velocity and $G\equiv G(x,y)$ is the particle flux function. Also, from Eq. (4) we find an estimate $v_{\perp}\sim\beta v_{\parallel}$. Then, neglecting the terms $\sim\beta^j$ with $j\geq 3$ we find, from Eqs. (1)–(3) and (7),

$$w\nabla\psi + V_E\nabla\varphi = \nabla G/n \tag{8}$$

($w=\beta v_{\parallel}$), from Eqs. (2) and (6)

$$-\nabla\varphi + (1+\alpha)\nabla T + T\nabla\ln(n/n_0) = K\nabla\psi \tag{9}$$

[$K\equiv K(x,y)$, and n_0 is a normalization constant] and from Eqs. (2), (3), (5), and (8)

$$(\mathbf{e}_z\times\nabla G)\cdot\nabla w + \beta^2(\mathbf{e}_z\times\nabla\psi)\cdot\nabla(nT)/M = 0. \tag{10}$$

Notice that from Eqs. (8) and (9) we find

$$n\nabla w\times\nabla\psi = -\nabla\ln(n/n_0)\times\nabla G, \tag{11}$$

$$\nabla T\times\nabla\ln(n/n_0) = \nabla K\times\nabla\psi. \tag{12}$$

Let us analyze the implications of these equations before proceeding to their solution. Eq. (8) gives the coupling of the parallel and perpendicular $\mathbf{E}\times\mathbf{B}$ velocities: w and V_E, as required by the continuity of the flow; that is, an unbalanced transverse flow should give rise to a parallel flow. Such an unbalance arises whenever the divergence of $v_{\perp}$ in (3) is not zero which occurs when $\nabla\varphi\times\nabla n\neq 0$. In this case, this term can be obtained from Eq. (9), taking the cross product with ∇n and assuming that T is a functional of n; i.e. $T(n)$,

$$\nabla n\times\nabla\varphi = K\nabla n\times\nabla\psi. \tag{13}$$

From here we see that the nonvanishing of $\nabla\cdot V_{\perp}$ must produce a density function that is not constant on the magnetic flux surfaces, implying in turn that the pressure in not a surface quantity. This exemplifies that the coupling of V_E and w, causes P to vary along a magnetic field line.

III. SOLUTIONS FOR THE PLASMA PARAMETERS

In order to make Eqs. (8)–(11) more tractable, we find that it is useful that instead of x and y we use other variables. First we will use the variables G and ψ. Then, from Eqs. (8) and (9) we find

$$w = V_E\left\{(1+\alpha)\frac{\partial T}{\partial\psi} + T\frac{\partial\Lambda}{\partial\psi} - K\right\}, \tag{14}$$

$$V_E\left\{(1+\alpha)\frac{\partial T}{\partial G} + T\frac{\partial\Lambda}{\partial G}\right\} = -\frac{1}{n}, \tag{15}$$

where $\Lambda=\ln(n/n_0)$. From Eqs. (10)–(12) we have

$$\frac{\partial w}{\partial\psi} = \frac{\beta^2}{M}\frac{\partial(nT)}{\partial G}, \tag{16}$$

$$\frac{\partial w}{\partial G} = -\frac{\partial}{\partial\psi}\left(\frac{1}{n}\right), \tag{17}$$

$$\frac{\partial T}{\partial G}\frac{\partial\Lambda}{\partial\psi} - \frac{\partial T}{\partial\psi}\frac{\partial\Lambda}{\partial G} = \frac{\partial K}{\partial G}. \tag{18}$$

In order to further simplify our equations we will assume that $K=K_{\psi}\psi$, where K_{ψ}=const. Then from Eq. (18) it follows that the density n must be a function of the temperature T: $n=N(T)$. Next we notice that from Eqs. (16) and (17) we have

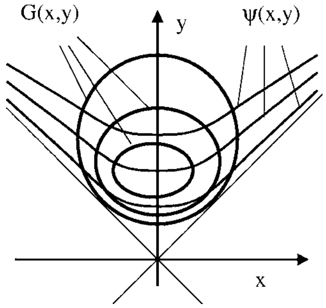

FIG. 1. Contours of constant magnetic flux $\psi(x,y)$ and constant perpendicular flow $G(x,y)$ near the X point that will give rise to a pressure variation over flux surfaces, consistent with observations in DIII-D.

$$\frac{\beta^2}{M}\frac{\partial^2(nT)}{\partial G^2}+\frac{\partial^2}{\partial\psi^2}\left(\frac{1}{n}\right)=0. \tag{19}$$

Therefore picking

$$N(T)\equiv n_0(T_0/T)^{1/2}, \tag{20}$$

from Eq. (19) we find

$$\frac{\beta^2 n_0^2 T_0}{M}\frac{\partial^2}{\partial G^2}(T/T_0)^{1/2}+\frac{\partial^2}{\partial\psi^2}(T/T_0)^{1/2}=0. \tag{21}$$

One of the possible solutions of Eq. (21) is

$$\left(\frac{T}{T_0}\right)^{1/2}=C_G G+C_\psi\psi, \tag{22}$$

where C_G and C_ψ are constants. Using Eqs. (20) and (22) from Eqs. (14)–(17) we find $C_G=1/V_E n_0 T_0(1+2\alpha)$ and the following relation between constants C_ψ and K_ψ

$$V_E K_\psi-(1+2\alpha)V_E T_0 C_\psi^2=\frac{\beta^2}{M}\frac{1}{(1+2\alpha)V_E}. \tag{23}$$

Thus, we find that for a strong coupling of parallel and perpendicular dynamics one may have equilibrium solutions with a large variation of plasma pressure on the magnetic flux surfaces:

$$\frac{T}{T_0}=\left(C_\psi\psi-\frac{G}{V_E n_0 T_0(1+2\alpha)}\right)^2, \tag{24}$$

$$\frac{n}{n_0}=\left(C_\psi\psi-\frac{G}{V_E n_0 T_0(1+2\alpha)}\right)^{-1}. \tag{25}$$

This equilibrium allows a qualitative explanation of the UEDGE modeling results, and of the experimental observation of the formation of high-density, cold plasma in the vicinity of the tokamak X point in DIII-D.[6] In this case we can, for example, take a magnetic flux function $\psi(x,y)\propto x^2-y^2$, and $G(x,y)$ in such a way as to have the equipotentials like those shown in Fig. 1, with $G(x,y)\to$const. far away from the X point ($x=y=0$). Such equilibrium has a strong variation of pressure, P, along the magnetic flux surfaces near the X point and a flux surface pressure, $P=P(\psi)$, far from it. This is shown in Fig. 2, where constant pressure contours are displayed (solid lines) along with magnetic contours (dashed lines) in the vicinity of the X point, and it is clearly appreciated that the two differ greatly just above this point, whereas there is a coincidence as the distance from the X point increases. Similar contours are found for the density and the temperature. Here we chose to place the region of circulation just above the X point instead of around it, to avoid dealing with the transitions between the four regions divided by the separatrix. The elliptic contours were obtained from the function $G(x,y)=[1+\{\sqrt{x^2+(y-y_0)^2}+\varepsilon(y-y_0)\}^{-1}]^{-1}$ which satisfies the condition $G(x,y)\to 1$ far from $(0,y_0)$.

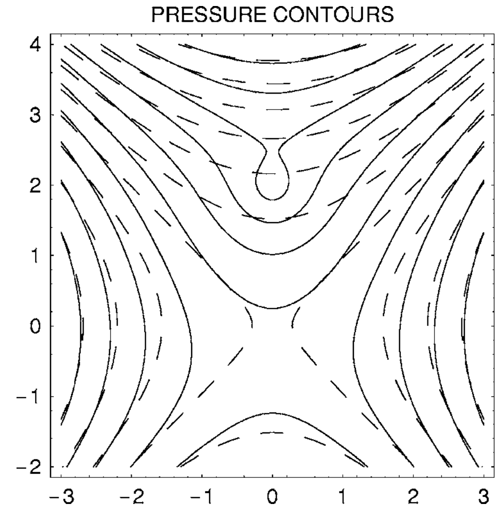

FIG. 2. Constant pressure contours obtained from Eqs. (24) and (25) for the functions $\psi(x,y)$ and $G(x,y)$ displayed in Fig. 1, showing that they differ from the $\psi(x,y)$=const. contours (dashed line) near the X point, but they coincide far from it. The maximum pressure is in the region where $G(x,y)$ is appreciable.

We also can use the variables $\xi=\ln(n/n_0)$ and ψ instead of x and y. Then, from Eqs. (8)–(10) we find

$$w-V_E(K-(1+\alpha)\,\partial T/\partial\psi)=(\partial G/\partial\psi)/n, \tag{26}$$

$$V_E(T+(1+\alpha)\,\partial T/\partial\xi)=(\partial G/\partial\xi)/n, \tag{27}$$

$$\frac{\partial G}{\partial\psi}\frac{\partial w}{\partial\xi}-\frac{\partial G}{\partial\xi}\frac{\partial w}{\partial\psi}+\frac{\beta^2}{M}\frac{\partial(nT)}{\partial\xi}=0, \tag{28}$$

and from Eqs. (11) and (12)

$$n\frac{\partial w}{\partial\xi}=-\frac{\partial G}{\partial\psi}, \tag{29}$$

$$\frac{\partial T}{\partial\psi}=-\frac{\partial K}{\partial\xi}. \tag{30}$$

Taking w from Eq. (26), $\partial G/\partial\xi$ from Eq. (27) and $\partial w/\partial\xi$ from Eq. (29) and substituting them in Eq. (28) we find

$$\frac{1}{n}\left(\frac{\partial G}{\partial\psi}\right)^2+nV_E\left(T+(1+\alpha)\frac{\partial T}{\partial\xi}\right)\times\left\{\frac{1}{n}\frac{\partial^2 G}{\partial\psi^2}-V_E\left((1+\alpha)\frac{\partial^2 T}{\partial\psi^2}-\frac{\partial K}{\partial\psi}\right)\right\}-\frac{\beta^2}{M}\frac{\partial(nT)}{\partial\xi}=0, \tag{31}$$

whereas from Eqs. (26) and (29) we have

$$\frac{1}{n}\frac{\partial^2 G}{\partial\xi\,\partial\psi} = -V_E\frac{\partial}{\partial\xi}\left(K-(1+\alpha)\frac{\partial T}{\partial\psi}\right). \tag{32}$$

So that, Eqs. (26), (27), (30), and (31) determine the equilibrium for this set of the variables. Taking $T=T(n)$ from Eq. (30) we find $K\equiv K(\psi)$. We take

$$K=K_\psi\psi, \quad G\equiv G_\psi\psi+G_\xi(\xi), \tag{33}$$

where K_ψ and G_ψ are some constants and $G_\xi(\xi)$ is some function of ξ. In this case Eqs. (26), (27), (29), and (32) are satisfied and from Eq. (31) we have

$$\frac{G_\psi^2}{n}+nV_E^2K_\psi\left(T+(1+\alpha)\frac{\partial T}{\partial\xi}\right)-\frac{\beta^2}{M}\frac{\partial(nT)}{\partial\xi}=0. \tag{34}$$

From Eq. (34) we find

$$T=\theta(n)\equiv\frac{a}{(1+2\alpha-b)n^2}+\frac{C_T}{n^{(1-b)/(1+\alpha-b)}}, \tag{35}$$

where $a=G_\psi^2/(V_E^2K_\psi)$, $b=\beta^2/(MV_E^2K_\psi)$, and C_T is a normalization constant. Examining the solution (33) and (35) we find that this is the extension of the results of Ref. 9 to an arbitrary magnetic geometry described by the flux potential ψ.

A third possibility to consider is to take ξ and G as the independent variables instead of x and y. The same analysis as before can be made in order to find the relevant relationships among the dependent functions of G and ξ. It is found that, if one again assumes that the temperature is a function of density only, $T=T(n)$, and $\psi(G,\xi)$ is assumed to be separable as $\psi(G,\xi)=\psi_G G+\psi_\xi(\xi)$, then the same solution (35) is recovered, with the condition $\psi_\xi(\xi)=G_\xi(\xi)$. This shows that this solution can be derived from different approaches, although with the last approach it is not evident that n and T are variable along the ψ constant surfaces because G is used instead of ψ.

IV. DISCUSSION AND CONCLUSIONS

We have studied the equilibrium states of an axisymmetric plasma under a cylindrical approximation using a two-fluid description, when there is a magnetic field that is predominantly in the "toroidal" direction. With this approach it was possible to obtain some analytical solutions for particular equilibria. The applicability of these solutions is specially important to certain regions of a toroidal plasma that satisfies the condition $|B_\perp|/B_z\sim\beta\ll 1$. The inner region (or high field side) of a tokamak where MARFEs are observed to appear is more prone to satisfy that condition, and therefore our results lend themselves to explain the establishment of an equilibrium state with a strong temperature decrease towards the radiating zone. More importantly, this analysis is most relevant to describe the region around the X point of the separatrix in a divertor, as there the poloidal field vanishes and the toroidal component is the main one. As we already mentioned earlier, the solution (24) and (25) found in the first case studied will be suitable for describing the observed circulation flow and maximum pressure observed about the X point in DIII-D.[6] For this case the velocity function $G(x,y)$ would have a circulatory pattern as the one shown in Fig. 1, and the corresponding pressure would be given by the product of Eqs. (24) and (25). It will present a maximum value near the point on maximum $G(x,y)$ (roughly near the center of the closed contours) and then decrease along flux surfaces. Far from this region the flow tends to vanish, indicated by the chosen condition that $G(x,y)\to$const. and thus the pressure returns to be a flux surface quantity $P(\psi)$ as in the usual static equilibrium.

In the governing equations considered we have neglected cross-field anomalous transport under the assumption that the transport time scale is much longer than the circulation time associated with the plasma flows. This condition is fulfilled for the case of the exchange time, τ_x, measuring the transit of the flow along an equipotential surface, about the X point in DIII-D, for the estimated value given in Ref. 6 is $\tau_x=0.3$ ms, which is smaller than the diffusion time $t=(L^2/D_e)\sim(4\text{ cm})^2/(10^4\text{ cm}^2/\text{s})\sim 2$ ms.

The establishment of circulation can be traced back to the shearing of the $\mathbf{E}\times\mathbf{B}$ flow. This fact was pointed out in Ref. 9, and for the case of DIII-D one may check that the magnitude of the relevant quantities confirms the claim. Indeed, an estimate of the $\mathbf{E}\times\mathbf{B}$ shear gives $V_E'\sim V_E/L\sim(e\varphi/T)(cT/eB)/L^2\sim 16D_B/L^2$, where D_B ($\sim10^4$ cm^2/s) is the Bohm diffusion coefficient; so again for $L\sim 4$ cm, $V_E'\approx(2\times10^{-4}\text{ s})^{-1}\sim\tau_x^{-1}$, which indicates that the exchange time for the flows coincides with the shearing.

The solution given in (35) is the same as the one found in Ref. 9 where the plasma flow in the divertor was studied. The explicit role played by the coupling of the parallel flow to the $\mathbf{E}\times\mathbf{B}$ flow in producing the pressure variation along the magnetic field was demonstrated. It was shown that the regime of detached divertor could be established by this effect. It is significant that we recovered this solution, meaning that it is applicable in more general situations, other than the divertor region.

The existence of the equilibrium states found here is quite interesting and it has experimental support. The next thing to ask regarding the plasma dynamics would be how stable these states are with respect to different perturbation modes. In the context of ideal MHD it would be interesting to study for instance the ballooning modes when there is a variable pressure along B. In relation to the flows about the X point, it would be necessary to consider the resistive X-point modes,[11] which are a type of resistive ballooning modes that can give rise to turbulence and, consequently, transport.

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