QUANTUM GRAVITY SIGNALS FROM ALGEBRAIC STABILITY CONSIDERATIONS

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ABSTRACT

We apply Lie algebra deformation theory to the problem of identifying the stable form of the quantum relativistic kinematical algebra. Three possible deformations are found, which introduce dimensionful constants. We also argue that, instead of positions, moments should serve as Lie algebra generators, leading to a radically different interpretation of the nature of the deformations.

Key words: Poincaré algebra deformations; Planck scale physics; invariant length scale.

1. INTRODUCTION

In the hope of exorcising some of the pathological aspects of quantum field theory and, more recently, of identifying an algebraic signature of quantum gravity, the idea of endowing spacetime with a noncommutative nature has been, over the years, actively pursued. A fairly direct approach in this direction is to consider possible deformations of the underlying kinematical Lie algebra (which codifies the nature of spacetime), led by the reasonable criterion of stability (see, e.g., (7; 1)). In this paper we determine the stable form of standard quantum relativistic kinematics. We find three possible deformations, the physical implications of which depend critically on the identification of the generators sitting opposite the $P$’s in the Heisenberg commutator — we call them $Z$’s here, and argue against their universally accepted interpretation as position operators. Furthermore, by identifying them with the moment operators, we show that spacetime noncommutativity is not an inevitable feature of stability, contrary to previous claims.

2. LIE ALGEBRA DEFORMATIONS AND THE CONCEPT OF STABILITY

In this section we briefly review the standard Lie algebra deformation theory that will be used in the rest of the paper. Relevant references are the original source for this material (8; 9), and (7), the last one also our main motivation to follow the stability path. For background on Lie algebra and group cohomology see (4). We deal throughout with finite-dimensional real Lie algebras.

A Lie algebra $G = (V, \mu)$ is constructed by providing a vector space $V$ with a bilinear antisymmetric product $\mu : V \times V \to V$, which satisfies the Jacobi identity. Given a basis $\{T_A\}$, $A = 1, \ldots, n$ for $V$, the algebra $G$ can be specified by its structure constants, $\mu(T_A, T_B) = i f_{AB}^C T_C$. Consequently, the set $L_n$ of $n$-dimensional real Lie algebras can be represented geometrically as a hypersurface embedded in $\mathbb{R}^N$ (with $N = n^2(n-1)/2$) where each $f_{AB}^C, A < B$, runs along an axis, and where the Jacobi identities provide the defining algebraic relations. The coordinates of a point $P$ of $L_n$ give the structure constants of the Lie algebra $G_P$.

$GL(N, \mathbb{R})$ acts on $L_n$ via linear redefinitions of the generators, $T'_A = M_A B T_B, M \in GL(N, \mathbb{R})$, under which the structure constants transform as

$$f'_{AB}^C = M_A^R M_B^S (M^{-1})^C_U f_{RS}^U,$$

and $P$ moves to $P_M$ — the corresponding algebras are isomorphic. The crucial observation is that there are two types of algebras in $L_n$: those that are completely surrounded by isomorphic algebras and those whose neighborhoods include non-isomorphic ones, called stable (or rigid) and unstable, respectively.

In physics, structure constants of Lie algebras are often identified with experimentally determined fundamental constants of a theory, e.g., $\hbar$ in Heisenberg’s commutator, or $c^{-2}$ in the Lorentz algebra. The experimental errors involved render the position of the corresponding point in $L_n$ uncertain. If the algebra employed is unstable, the physical predictions of the theory become ill defined, as they depend critically on the (unknown) exact values of the structure constants. If, however, the algebra is stable,
ial deformations are generated by 2-coboundaries. Geometrically, the product among cochains is the 2-th cohomology group of the space, the corresponding vector spaces being denoted by $C^0(V)$ isomorphic to $V$. Also, 0-cochains are constant maps is an easy derivation of the equation for a 2-cocycle $\mu$, and the action of $s_\mu$ on an arbitrary $(p + 1)$-cochain $\psi \in \text{Alt}^p(V)$ is given by

$$s_\mu \triangleright \psi = (-1)^p [\mu, \psi] \equiv (-1)^p D_\mu \psi,$$

where the second equality defines the operator $D_\mu \equiv [\mu, \cdot]$. A useful property of $D_\mu$ is that it is a graded derivation in $V$.

$$D_\mu [\alpha, \beta] = [D_\mu \alpha, \beta] + (-1)^m [\alpha, D_\mu \beta],$$

where $\alpha \in \text{Alt}^m(V)$ and $\beta \in \text{Alt}(V)$. Eq. (7) allows an easy derivation of the equation for finite deformations. If $\mu$ is a Lie product, $\mu' = \mu + \phi$ will also be one if $[\mu', \mu'] = 0$, from which one gets immediately the deformation equation

$$D_\mu \phi + \frac{1}{2} [\phi, \phi] = 0,$$

which reduces to the cocycle condition for infinitesimal $\phi$.

Given a Lie algebra $\mathcal{G} = (V, \mu)$ and a deformation $\mu_t = \mu + \phi_t$, where $\phi_t = \sum_{n=1}^{\infty} \phi_n t^n$, substituting $\phi_t$ in (8) results in a series of equations for the $\phi_n$, one for each power of $t$. The equations corresponding to $t$, $t^2$ and $t^3$, are

$$D_\mu \phi_1 = 0, \quad D_\mu \phi_2 = -\frac{1}{2} [\phi_1, \phi_1], \quad D_\mu \phi_3 = -[\phi_1, \phi_2].$$

The first of (9) says that $\phi_1$ is a 2-cocycle. Then the graded derivation property of $D_\mu$ implies that $[\phi_1, \phi_1]$ is a 3-cocycle. The second of (9) may be solved for $\phi_2$ provided that this 3-cocycle be a coboundary, which may not be the case if $H^2(V, D_\mu)$ is non-trivial. We conclude that the existence of non-trivial 3-cocycles may render infinitesimal deformations non-integrable. If $[\phi_1, \phi_1]$ is indeed a trivial 3-cocycle, so that the second of (9) admits
a solution, an obstruction may occur in the next step, i.e., in the third of (9), and so on. It can be shown that all of these obstructions lie in $H^3$, so that, if $H^3$ is trivial, every non-trivial 2-cocycle is the first order term of some finite deformation (9). Referring back to our geometrical image of $L_n$, as a hypersurface in $\mathbb{R}^N$, non-integrable 2-cocycles correspond to deformation directions that point outside of $L_n$ but such that, for a little step of order $t$ along them, the Jacobi identities are violated to order $t^3$, or higher.

If $\phi$ is a (non-trivial) 2-cocycle satisfying $[\phi, \phi] = 0$, then Eq. (8) implies that $\mu + t\phi$, for $t$ finite, is a Lie product, if $\mu$ is one. When the dimension of $H^2$ is greater than one, the vanishing of the anticommutators $[\phi_t, \phi_j]$, $\phi_t, j \in H^2$, turns the finite deformation space of $G$ into a vector space, since an arbitrary linear combination $\phi$ of the cocycles also satisfies Eq. (8). Notice that a non-trivial 2-cocycle satisfying $[\phi, \phi] = 0$ leads to non-isomorphic algebras infinitesimally, but when extended to a finite deformation it may well lead, for particular values of $t$, to isomorphic algebras — we will encounter such a case in Sect. 3 below.

It is obvious from the definition given above, that a p-cochain can be realized as a $G$-valued left invariant (LI) p-form on the group manifold $G$ corresponding to $G$, with the generators $T_A$ now extended to LI vector fields. Denoting by $\{\Pi^A\}$ the LI 1-forms on $G$ dual to the generators $\{T_B\}$,

$$\langle \Pi^A, T_B \rangle = \delta_B^A,$$

we write $\psi^{(p)}$ as

$$\psi^{(p)} = \psi^B \otimes T_B = \frac{1}{p!} \psi_{A_1 \ldots A_p} B \Pi^{A_1} \ldots \Pi^{A_p} \otimes T_B.$$  

(10)

Then the action of $s$ given in (18) coincides with that of an exterior covariant derivative $\nabla$,

$$\nabla (\psi^A \otimes T_A) = (d\psi^A + \Omega^A_B \psi^B) \otimes T_A,$$

(12)

with the connection 1-form $\Omega$ given by

$$\Omega^A_B = f_{RB}^A \Pi^R,$$

(13)

where undotted indices in forms refer to $P$'s and dotted ones to $Z$'s, so that, e.g., $\langle \Pi^\mu, Z^\nu \rangle = \delta_\mu^\nu$. Slightly abusing notation, we will let $\Pi^M$ denote the 1-form that detects the generator $M$.

3. STABLE QUANTUM RELATIVISTIC KINEMATICS

In this section we engage on the search of a stable Lie algebra containing relativistic and quantum effects. Our starting point will be the fifteen-generator algebra $G_{PH}(q)$ (for “Poincaré - Heisenberg”),

$$[J_{\mu\nu}, J_{\rho\sigma}] = i \left( g_{\mu\sigma} J_{\nu\rho} + g_{\nu\rho} J_{\mu\sigma} - g_{\mu\rho} J_{\nu\sigma} - g_{\nu\sigma} J_{\mu\rho} \right)$$

(14)

$$[J_{\rho\sigma}, P_\mu] = i \left( g_{\rho\mu} P_\sigma - g_{\sigma\mu} P_\rho \right)$$

(15)

$$[J_{\rho\sigma}, Z_\mu] = i \left( g_{\rho\mu} Z_\sigma - g_{\sigma\mu} Z_\rho \right)$$

(16)

$$[P_\mu, Z_\nu] = i q g_{\mu\nu} M,$$

(17)

where $J_{\mu\nu}$ are the generators of the Lorentz group, $P_\mu$ are the momenta, $Z_\mu$ are generally identified with the positions (an interpretation we will soon challenge) and $M$ is a central generator whose only function in the literature is to render the r.h.s. of the (covariant form of the) Heisenberg commutator, Eq. (17), linear in the generators — to our knowledge, its physical nature has never been clarified. Its origin, which is known to occur, leads to spurious non-linearities forced by the Jacobi identities. This happened in the first work to deal with non-commuting spacetime coordinates, (11), and was pointed out in (12), with the story repeating itself almost sixty years later in (5), (2), respectively. For the moment, we regard $G_{PH}(q)$ as an abstract Lie algebra, devoid of any physical connotations, and inquire about its stability. Related works are (7; 6).

The 2-cochain $\mu_{\alpha}(q)$, corresponding to $G_{PH}(q)$, is given by

$$\mu_{\alpha}(q) = \frac{1}{2} \Pi^{\alpha\beta} \Pi^\beta_\alpha \otimes J_{\alpha\beta} + \Pi^{\alpha\beta} \Pi_\alpha \otimes P_\alpha + \Pi^{\alpha\beta} \Pi_\beta \otimes Z_\alpha + q \Pi^{\alpha\beta} \Pi_\alpha \otimes M,$$

(18)

where unkotted indices in forms refer to $P$'s and dotted ones to $Z$'s, so that, e.g., $\langle \Pi^\mu, Z_\nu \rangle = \delta_\mu^\nu$. Slightly abusing notation, we will let $\Pi^M$ denote the 1-form that detects the generator $M$.

We find that $H^2(G_{PH}(q))$ is non-trivial (see (3) for the details of the calculations on this section),

$$H^2(G_{PH}(q)) = \{[0], [\zeta_1], [\zeta_2], [\zeta_3] \},$$

(19)

where

$$\zeta_1 = \Pi^{\mu\nu} M \otimes Z_\mu + \frac{q}{2} \Pi^{\mu\nu} \otimes J_{\mu\nu}$$

(20)

$$\zeta_2 = -\Pi^{\mu\nu} M \otimes P_\mu + \frac{q}{2} \Pi^{\mu\nu} \otimes J_{\mu\nu}$$

(21)

$$\zeta_3 = \Pi^{\mu\nu} M \otimes Z_\mu - \Pi^{\mu\nu} M \otimes P_\mu + q \Pi^{\mu\nu} \otimes J_{\mu\nu}.$$  

(22)

We also find that all anticommutators among the $\zeta$'s vanish. Accordingly, an arbitrary linear combination $\zeta(\vec{a}) = \sum \alpha_i \zeta_i$ (sum over $i$ implied), for finite $\alpha_i$, provides the finite deformation $G_{PH}(q, \vec{a})$ of $G_{PH}(q)$. The deformed commutators are

$$[P_\mu, Z_\nu] = i q g_{\mu\nu} M + i q \alpha_3 J_{\mu\nu}$$

(23)

$$[P_\mu, P_\nu] = i q \alpha_1 J_{\mu\nu}$$

(24)

$$[Z_\mu, Z_\nu] = i q \alpha_2 J_{\mu\nu}$$

(25)

$$[P_\mu, M] = -i \alpha_3 P_\mu + i \alpha_1 Z_\mu$$

(26)

$$[Z_\mu, M] = -i \alpha_2 P_\mu + i \alpha_3 Z_\mu,$$

(27)
complemented by those in Eqs. (14)–(16). For a generic deformation, the $P$’s cease to commute among themselves, the same happens with the $Z$’s, $M$ is no longer central, while the Heisenberg commutator receives an additional term, proportional to $J_{\mu\nu}$.

Is $G_{PH}(\alpha, \bar{\alpha})$ stable? We compute, again, the second cohomology group and find

$$H^2(G_{PH}(\alpha, \bar{\alpha})) = \begin{cases} \{0\} & \text{if } \alpha_2^2 \neq \alpha_1 \alpha_2 \\ \{0, [\chi]\} & \text{if } \alpha_3^2 = \alpha_1 \alpha_2 \end{cases},$$

where $\chi = \zeta_1 + \zeta_2$ satisfies $[\chi, \chi] = 0$. $G_{PH}(\alpha, \bar{\alpha})$ is, accordingly, stable everywhere outside the instability surface $\alpha_3^2 = \alpha_1 \alpha_2$ in $\alpha$-space. The latter represents a double cone with the apex at the origin and its axis along the first diagonal in the $\alpha_1$-$\alpha_2$ plane, parallel to $\chi$ (see Fig. 1). We will refer to the various regions of $\alpha$-space with their relativistic nicknames (“future”, “past”, “etc.”), with the positive $\alpha_1 \alpha_2$ quadrant lying in the future. It is easily shown that there are six equivalence classes of algebras, given by the regions of the $\alpha$-space is divided into by the double light cone: future, past, elsewhere, future cone, past cone, apex. For each of the above classes, a representative exists with $\alpha_3 = 0$. An arbitrary point in $\alpha$-space may be brought on the $\alpha_1 \alpha_2$ plane by a rotation in the $P_\mu, Z_\mu$ planes, $P'_{\mu} = \cos(\theta)P_{\mu} + \sin(\theta)Z_{\mu}$, $Z'_{\mu} = -\sin(\theta)P_{\mu} + \cos(\theta)Z_{\mu}$, which rotates $\alpha$-space by an angle $2\theta$ around the axis of the cone, counterclockwise as seen from the future. The $\alpha_i$, $i = 1, 2, 3$, are fundamental constants of the theory of possibly Planckian and/or cosmological origin (see, e.g., (5)).

As it has been pointed out in (6; 7), off the instability cone, $G_{PH}(\alpha, \bar{\alpha})$ is isomorphic to some $so(m, 6 - m)$, where, taking $\alpha_3 = 0$, $m$ depends on the signs of $q$, $\alpha_1$ and $\alpha_2$. Specifically, assuming $q > 0$,

$$G_{PH}(q, \alpha_1, \alpha_2, \alpha_3 = 0) \cong \begin{cases} so(1, 5) & \text{if } \alpha_1 > 0, \alpha_2 > 0 \\ so(2, 4) & \text{if } \alpha_1 \alpha_2 < 0 \\ so(3, 3) & \text{if } \alpha_1 < 0, \alpha_2 < 0 \end{cases}.$$  

4. SOME PHYSICAL CONSIDERATIONS

We deal, finally, with a number of interpretational issues. First, we wish to discuss the physical meaning of the coproduct of Lie algebra generators. We use the Poincaré algebra as an example, but the discussion applies to general Lie algebras.

Consider the state of a particle which is represented by a state vector $|\psi\rangle$ in some Hilbert space $\mathcal{H}$. To a possible transformation of the system, e.g., a rotation $R_{\alpha_3 \beta_3}$ parametrized by Euler’s angles, one associates an operator $D(R_{\alpha_3 \beta_3})$, acting on $\mathcal{H}$. Imagine now that, under closer inspection, the particle is seen to be a bound state of two other particles, say, particles 1 and 2. The state space becomes $\mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_i$ is the state space of particle $i$. To rotate what is now known to be a two-particle system, one applies the above rotation to each of the constituent particle systems. This observation implies that the operator representing $R_{\alpha_3 \beta_3}$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is simply $D_1(R_{\alpha_3 \beta_3}) \otimes D_2(R_{\alpha_3 \beta_3})$, where $D_i$ is the representation of rotations in $\mathcal{H}_i$. This is true for all representations $D_i$ — we may accordingly conclude that the abstract rotation operator $R_{\alpha_3 \beta_3}$ acts on tensor products as $R_{\alpha_3 \beta_3} \otimes R_{\alpha_3 \beta_3}$ and call this latter operator the coproduct $\Delta(R_{\alpha_3 \beta_3})$ of $R_{\alpha_3 \beta_3}$. The fact that rotations should compose in the same way, whether applied to a simple or to a composite system, is expressed algebraically by the requirement that

$$\Delta(R_1 R_2) = \Delta(R_1) \Delta(R_2).$$

Summarizing, for all transformations $S$ in the Poincaré group, the coproduct $\Delta(S)$ is grouplike,

$$\Delta(S) = S \otimes S,$$  

and $\Delta$ is an algebra homomorphism,

$$\Delta(S_1 S_2) = \Delta(S_1) \Delta(S_2).$$

Now write $S = e^A$, with $A$ in the Poincaré algebra $g_P$ and define $\Delta$ to be linear in the entire $U(g_P)$, the universal enveloping algebra of $g_P$ — a simple calculation then shows that $\Delta(A) = A \otimes 1 + 1 \otimes A$ (this is a logarithm turning a product into a sum, as usual). We conclude that the generators of grouplike transformations are primitive,

$$\Delta(A) = A \otimes 1 + 1 \otimes A,$$  

with $J_{tot} = J_1 + J_2$ as the archetypal example from quantum mechanics. In other words, the physical quantities corresponding to generators of grouplike transformations are additive under system composition. All Lie algebra

\[3\] More precisely, a certain topological completion of $U(g_P)$.
generators are of this nature. As a conclusion, only primitive operators should be allowed as Lie algebra generators.

We have seen that all the generators of the Poincaré algebra satisfy the primitiveness condition, but what about the position operators \(X_\mu\)? The answer is that \(X_\mu\) is not primitive. To begin with, it is rather obvious that position is not an extensive quantity: if two particles are glued together at \(x_\mu\), their composite system is also located at \(x_\mu\), not at \(2x_\mu\). At the finite level, where the position operators can be regarded as generators of translations in momentum space, it is clear that translating each of the two particles forming a composite system by \(k\) in momentum space, one ends up with the composite particle being translated by \(2k\), not \(k\). Either way, it becomes evident that the position operators are not primitive and, hence, cannot be taken as generators of a Lie algebra. At the finite level, \(e^{ik\cdot X_\mu}\) cannot serve as points on the group manifold. This last consideration shows something about the nature of the grouplike operator that should replace \(e^{ik\cdot X_\mu}\), the logarithm of which would be acceptable as a Lie algebra generator. Roughly speaking, it should somehow translate the particle in momentum space by a quantity proportional to its mass.

So, if \(X_\mu\) is not primitive, what is its coproduct \(\Delta(X_\mu)\)? The answer requires to clarify what we are going to ask the position operator to do. For a single localized particle it is clear that \(X_\mu\) should return its position, but what should \(X_\mu\) (via its coproduct \(\Delta(X_\mu)\)) do on a two-particle system? Clearly, if the two particles are glued together and the composite system is localized, we should get the same answer whether we operate with \(X_\mu\) on the composite particle or with \(\Delta(X_\mu)\) on the two-particle system. When the two particles are far apart and/or have different velocities, the natural demand would be that \(\Delta(X_\mu)\) should return the position of their center-of-momentum, which is the natural “effective position” of a relativistic composite system\(^4\). The problem now is that there exist various proposals for the definition of the “effective position” of a composite relativistic system, some of which imply that the latter does not behave as a 4-vector. In those cases, different observers locate the center-of-momentum of a system at different points. At the algebraic level, this means that \(\Delta\) fails to be a homomorphism of the \(J\)-\(X\) commutation relations, in other words, the coproduct of the \(X\)'s, in the above cases, does not even exist. For the definitions of the center-of-momentum that do yield a 4-vector, the coproduct can be read off from the expression for the center-of-momentum coordinates of a two-particle system in terms of those of the constituent particles.

In either case though, we know from experience that there are composite systems the position of which, at least approximately, behaves like a 4-vector, e.g., butterflies. This implies that even though \(\Delta(X_\mu)\) may not exist, depending on the definition of the center-of-momentum, it is possible to define an approximate coproduct that works on a restricted class of systems — intuitively, systems that can fool the observer into believing that they are a single, localized particle. As an example, consider the definition for the center-of-momentum of a (non-interacting) two-particle system given by

\[
\vec{R} = \frac{E_1 \vec{r}_1 + E_2 \vec{r}_2}{E},
\]

where \(E \equiv E_1 + E_2\) is the total energy of the system. Clearly, \(\vec{R}\) is not part of a 4-vector. Next, consider a system such that in its center-of-momentum frame all energies \(E_i\) are nearly equal to the corresponding rest masses, \(E_i \approx m_i\) — we call such systems psychron, from the greek word for “cold”. For such a system (33) reduces to the Newtonian formula for the center-of-mass. Moreover, boosting to an arbitrary frame, all energies rescale by the same \(\gamma\)-factor, which cancels, so the l.h.s. does transform as a spatial vector. We conclude that, for psychron 2-particle systems, the relation

\[
m_{12} x^\mu_{12} = m_1 x^\mu_1 + m_2 x^\mu_2,
\]

where \(m_{12} \equiv m_1 + m_2\), defines the effective position \(x_{12}\) of the system as a 4-vector. So, if we denote by \(M\) the mass operator \((M^2 = P^\mu P_\mu)\), we conclude from (34) that the moment operator \(Z_\mu = X_\mu M\) is primitive when applied to psychron systems.

To investigate the repercussions of the above interpretation of \(Z_\mu\), consider the standard quantum relativistic algebra \(\mathcal{G}_{QR}\), where the \(P\)'s and the \(X\)'s commute, and the cross-relations are given by the Heisenberg commutator. Then the \(Z\)-\(Z\) and the \(Z\)-\(M\) commutation relations are fixed, and there is no \textit{a priori} reason why they should close linearly in the \(\{J, P, Z, M\}\) set. Nevertheless, we find

\[
[Z_\mu, Z_\nu] = i q (X_\mu P_\nu - X_\nu P_\mu), \quad [Z_\mu, M] = -i q P_\mu.
\]

where \(\nu \neq 0\), with no connection to spacetime non-commutativity. We identify the r.h.s. of the first equation as a multiple of the orbital angular momentum operator, \(L_{\mu\nu} = q^2 (X_\mu P_\nu - X_\nu P_\mu)\). Then, for a massive, spinless particle,

\[
[Z_\mu, Z_\nu] = i q^2 J_{\mu\nu}.
\]

A look at (25) and (27) shows that the above relations are of exactly the form of those provided by the \(\alpha_2\) deformation, with \(\alpha_2 = q\). This means that \(\mathcal{G}_{QR}\), the algebra of standard quantum relativistic kinematics lies at the point \(\alpha = 0\) in \(\alpha\)-space, having only one non-trivial deformation available (along \(\chi\)), which introduces non-commutativity among the momenta. The above argument shows that spacial non-commutativity, cannot of course be ruled out, but is not an inevitable feature of stabilization.

\(^4\)See, for example, the discussion in (10), p. 84.
5. CONCLUDING REMARKS

We have taken in this paper the stability criterion to its ultimate consequences. Our systematic algebraic analysis has provided a general map of the possible deformations, establishing the uniqueness of previous results and shedding light on various technical issues, in particular, the interrelations among the deformations found. A fundamental departure from the standard picture has been our identification of the $Z_\mu$ generators with the moment operators of a (massive, spinless) particle, having concluded that the position operators lack the essential property of primitiveness, necessary for all Lie algebra generators.

We think that this work suggests some directions of study that deserve further consideration. First, we would like to generalize the concept of the moment operators to the case of particles with spin, and/or zero mass. Second, representation theoretical aspects of the problem should be examined, in particular, a Wigner-type classification should be carried through. It would also be of interest to develop some degree of intuition regarding the deformed kinematics, e.g., by clarifying the coexistence of the Lorentz contraction with an invariant length scale.

The hope is that, after two impressive (albeit a posteriori) vindications of the stability criterion, in the form of the relativistic and quantum revolutions, some true predictions might await us further ahead the deformation path.

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