Beauty
in Physics
Cocoyoc, 14—18 May 2012

LINEAR CANONICAL TRANSFORMS:
40 years of developments

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Back in 1970…

the same formulas were found at the same time in two very different places:

Stuart A. Collins Jr. (Electroscience Lab, Ohio State University, Columbus) described light propagation in the **paraxial régime** through an integral kernel, built from thin lenses and empty spaces.

Lens-system diffraction integral written in terms of matrix optics
J. Opt. Soc. Am. **60**, 1168—1177 (1970)

Marcos Moshinsky and **Christiane Quesne** (Instituto de Física, UNAM) sought for the **conservation of uncertainty** under linear transformations of phase space, as a matter of intrinsic mathematical interest.

Oscillator Systems. In: Proceedings of the 15th Solvay Conference in Physics (1970)
Linear canonical transformations and their unitary representation
J. Math. Phys. **12**, 1772-1780, 1780—1783 (1971)

Stuart A. Collins Jr.

Professor Emeritus

Lens-System Diffraction Integral Written in Terms of Matrix Optics
J. Opt. Soc. Am. **60**, 1668-1177,1970

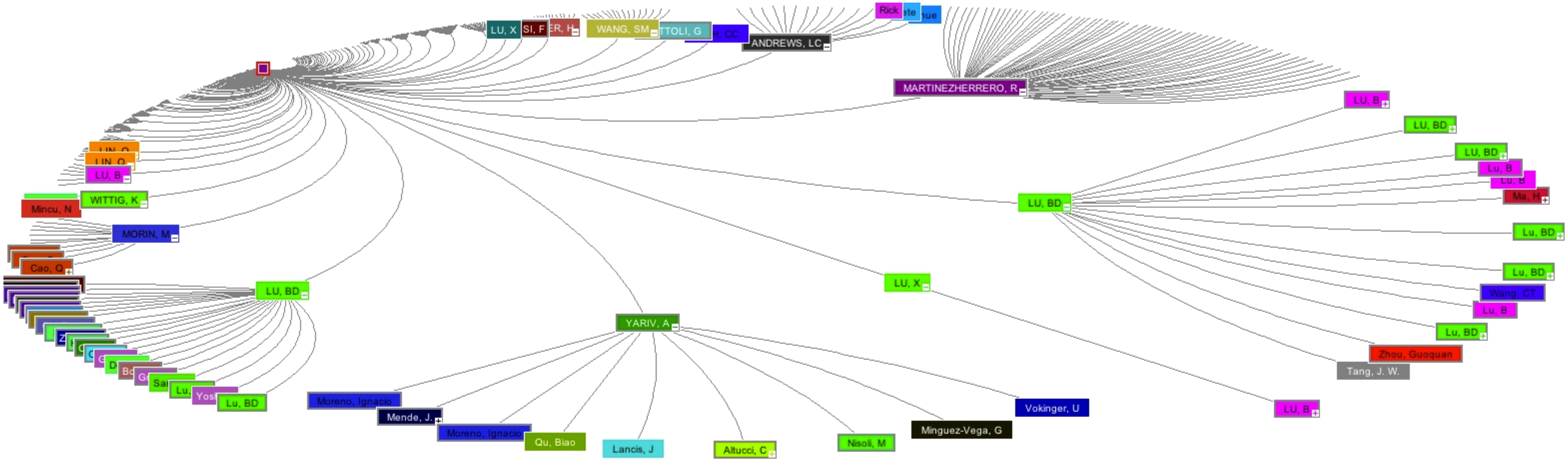
>514 citations (Feb 2012)

The ElectroScience Laboratory
The Ohio State University
Columbus, Ohio

Optoelectronics
Space Science

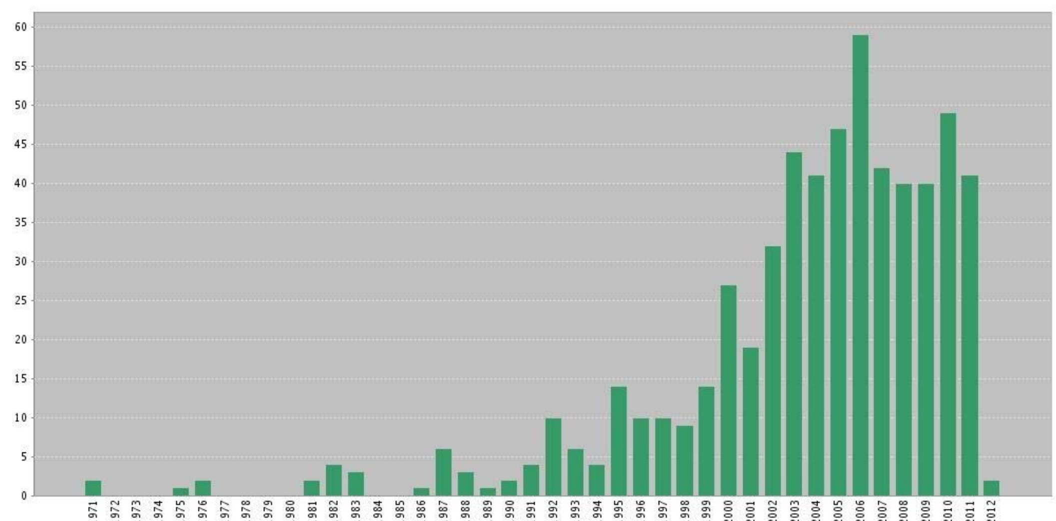
6 patents from 1982 to 2008

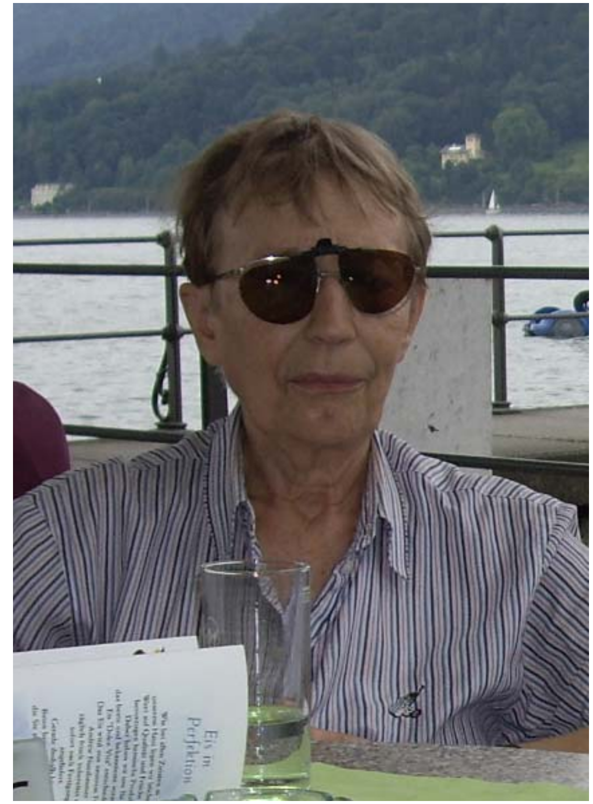
1st and 2nd –order citations to the paper of Stuart Collins



Source:Web of Knowledge™, www.thomsonscientific.com

citations per year





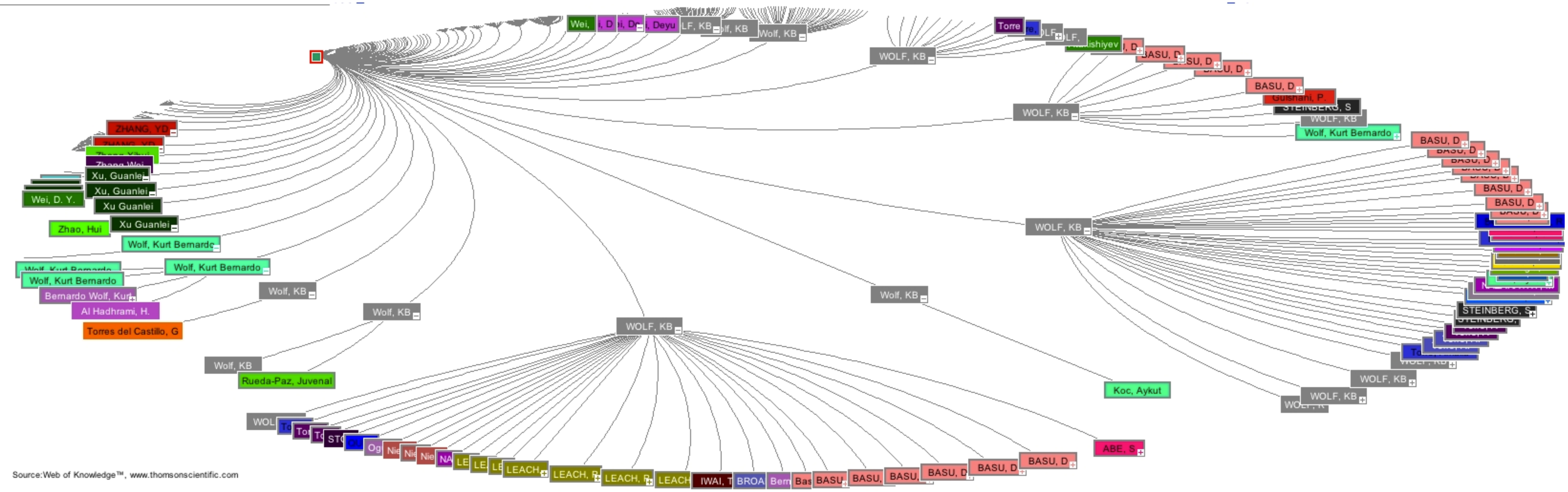
Marcos Moshinsky
Instituto de Física
Universidad Nacional Autónoma de México

Christiane Quesne
Faculté de Sciences
Université Libre de Bruxelles

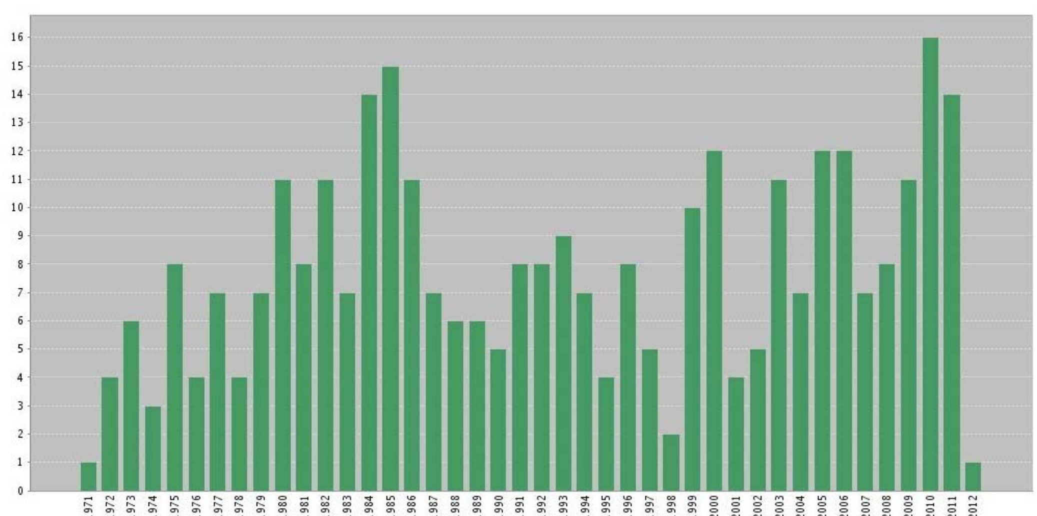
Oscillator Systems. In: Proceedings of the 15th Solvay Conference in Physics (1970)
Linear canonical transformations and their unitary representation
J. Math. Phys. **12**, 1772-1780, 1780—1783 (1971)

>346 citations (Feb 2012)

1st and 2nd –order citations to the paper of Moshinsky & Quesne



citations per year



Lens-system diffraction integral written in terms of matrix optics
J. Opt. Soc. Am. **60**, 1168—1177 (1970)

The paraxial optical construction with matrices

(Brower 1964, Gerrard & Burch 1975)

x: ray position, **p**: *n* x ray slope

$$\text{thin lens: } \mathcal{C}\begin{pmatrix}1 & 0\\ \tau & 1\end{pmatrix}, \quad \text{free flight: } \mathcal{C}\begin{pmatrix}1 & -\tau\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}\mathbf{x}'\\ \mathbf{p}'\end{pmatrix} = \begin{pmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & \mathbf{d}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{p}\end{pmatrix}, \quad \text{i.e.,} \quad \mathbf{w}' = \mathbf{M}\mathbf{w},$$

The **Fresnel** transform is integral, *ergo*:

$$f_{\mathrm{M}}(\mathbf{x}) \equiv (\mathcal{C}_{\mathrm{M}}\, f)(\mathbf{x}) := \int_{\mathsf{R}^N} \mathrm{d}^N\mathbf{x}'\, C_{\mathrm{M}}(\mathbf{x},\mathbf{x}')\, f(\mathbf{x}'),$$

1. Find the excess quadratic phase of off-axis rays in compound systems.
2. Find the Lagrangian that requires $\mathbf{a}\,\mathbf{b}^\top = (\mathbf{a}\,\mathbf{b}^\top)^\top, \quad \mathbf{c}\,\mathbf{d}^\top = (\mathbf{c}\,\mathbf{d}^\top)^\top, \quad \mathbf{a}\,\mathbf{d}^\top - \mathbf{b}\,\mathbf{c}^\top = \mathbf{1}.$

$$C_{\mathrm{M}}(\mathbf{x},\mathbf{x}') := K_{\mathrm{M}} \exp \mathrm{i}\Big(\tfrac{1}{2}\mathbf{x}^\top\mathbf{b}^{-1}\mathbf{d}\mathbf{x} - \mathbf{x}^\top\mathbf{b}^{-1}\mathbf{x}' + \tfrac{1}{2}\mathbf{x}'^\top\mathbf{a}\mathbf{b}^{-1}\mathbf{x}'\Big).$$

3. Find the phase (only from the Fresnel transform).
4. Find normalization from energy conservation.

Oscillator Systems. In: Proceedings of the 15th Solvay Conference in Physics (1970)
Linear canonical transformations and their unitary representation
J. Math. Phys. **12**, 1772-1780, 1780—1783 (1971)

The conservation of uncertainty

$$[\hat{x}_i, \hat{p}_j] := \hat{x}_i\hat{p}_j - \hat{p}_j\hat{x}_i = \mathrm{i}\,\delta_{i,j},$$

under linear transformations of $\mathbf{w} = \binom{\mathbf{x}}{\mathbf{p}}$, $\mathcal{C}_{\mathrm{M}}\,\hat{\mathbf{w}}\,\mathcal{C}_{\mathrm{M}}^{-1} = \mathbf{M}^{-1}\hat{\mathbf{w}}$.

Preservation of HW => $\mathbf{M}\,\boldsymbol{\Omega}\,\mathbf{M}^{\top} = \boldsymbol{\Omega}$, $\quad \boldsymbol{\Omega}^{\top} = -\boldsymbol{\Omega}$, $\quad \boldsymbol{\Omega}^2 = -\mathbf{1}$, $\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$.

symplectic conditions.

Integral transform form

$$f_{\mathrm{M}}(\mathbf{x}) \equiv (\mathcal{C}_{\mathrm{M}}\,f)(\mathbf{x}) := \int_{\mathsf{R}^N} \mathrm{d}^N\mathbf{x}'\,C_{\mathrm{M}}(\mathbf{x},\mathbf{x}')\,f(\mathbf{x}'),$$

acting on $\mathbf{x}f$ and on $\mathbf{p}f$ =>

2N differential equations

for the kernel:

$$x_i' C_{\mathrm{M}}(\mathbf{x},\mathbf{x}') = \textstyle\sum_j \left(d_{j,i}x_i + \mathrm{i}b_{j,i}\partial_j\right) C_{\mathrm{M}}(\mathbf{x},\mathbf{x}'),$$
$$\partial_i' C_{\mathrm{M}}(\mathbf{x},\mathbf{x}') = \textstyle\sum_j \left(\mathrm{i}c_{j,i}x_i - a_{j,i}\partial_j\right) C_{\mathrm{M}}(\mathbf{x},\mathbf{x}').$$

$$C_{\mathrm{M}}(\mathbf{x},\mathbf{x}') := K_{\mathrm{M}} \exp \mathrm{i}\left(\tfrac{1}{2}\mathbf{x}^{\top}\mathbf{b}^{-1}\mathbf{d}\mathbf{x} - \mathbf{x}^{\top}\mathbf{b}^{-1}\mathbf{x}' + \tfrac{1}{2}\mathbf{x}'^{\top}\mathbf{a}\mathbf{b}^{-1}\mathbf{x}'\right).$$

Limit to the identity implies $K_{\mathrm{M}} = \dfrac{1}{\sqrt{(2\pi\mathrm{i})^N \det \mathbf{b}}} \equiv \dfrac{e^{-\mathrm{i}\pi N/4} \exp \mathrm{i}(-\frac{1}{2}\arg\det\mathbf{b})}{\sqrt{(2\pi)^N |\det \mathbf{b}|}}$

The well-known 1-dim LCTs

$$f_{\mathrm{M}}(x) \equiv (\mathcal{C}_{\mathrm{M}} f)(x) = \int_{\mathsf{R}} \mathrm{d}x'\, C_{\mathrm{M}}(x,x')\, f(x'),$$

$$C_{\mathrm{M}}(x,x') := \frac{e^{\mathrm{i}\pi/4}}{\sqrt{2\pi\mathrm{i}\, b}} \exp\Big(\frac{\mathrm{i}}{2b}(dx^2 - 2xx' + ax'^2)\Big),$$

…be careful with phases: $1/\sqrt{\mathrm{i}b} = e^{-\mathrm{i}\pi(\operatorname{sign} b + \frac{1}{2})/2}/\sqrt{|b|}.$

LCTs in the Hilbert space $L^2(R)$ are unitary,

$$(f,g)_{\mathcal{L}^2(\mathsf{R}^N)} := \int_{\mathsf{R}^N} \mathrm{d}\mathbf{x}\, f(\mathbf{x})^*\, g(\mathbf{x}) = (f_{\mathrm{M}}, g_{\mathrm{M}})_{\mathcal{L}^2(\mathsf{R}^N)},$$

$$C_{\mathrm{M}}(\mathbf{x},\mathbf{x}') = C_{\mathrm{M}^{-1}}(\mathbf{x}',\mathbf{x})^*.$$

They compose as Sp(2,R)
--up to a metaplectic sign

$$\int_{\mathsf{R}^N} \mathrm{d}\mathbf{x}'\, C_{\mathrm{M}_1}(\mathbf{x},\mathbf{x}')\, C_{\mathrm{M}_2}(\mathbf{x}',\mathbf{x}'') = \sigma\, C_{\mathrm{M}_1\mathrm{M}_2}(\mathbf{x},\mathbf{x}''),$$

They include the Fourier transform
--but for a phase, so they cover
the Fourier cycle twice.

$$\mathcal{C}_{\mathrm{F}} = e^{-\mathrm{i}\pi/4}\mathcal{F}, \qquad \mathbf{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The **metaplectic** phase

has bedevilled countless paraxial optical papers…
because their authors did not realize that $\mathrm{Sp}(2N,R)$
has an <u>infinite</u> cover group,
of which the integral realization is the *double* cover.

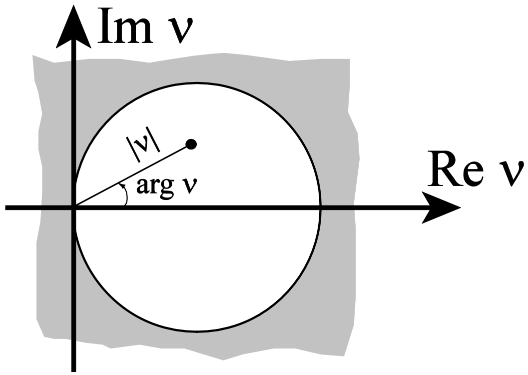
Valentin Bargmann
has shown that
if we use the polar
matrix decomposition
we can parametrize
any cover group (1947)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \lambda + \mathrm{Re}\,\mu & \mathrm{Im}\,\mu \\ \mathrm{Im}\,\mu & \lambda - \mathrm{Re}\,\mu \end{pmatrix}$$

$$\phi = \phi_1 + \phi_2 + \arg\nu, \qquad \mu = e^{-\mathrm{i}\arg\nu}(\lambda_1\mu_2 + e^{-2\mathrm{i}\phi_2}\mu_1\lambda_2),$$

$$: \nu := 1 + e^{-2\mathrm{i}\phi_2}\mu_1\mu_2/\lambda_1\lambda_2$$

$$\arg\nu \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi), \text{ and } \lambda = \lambda_1|\nu|\lambda_2 \geq 1.$$



…actually, this was realized several years later…
We were concerned with the harmonic oscillator
rather than with the fractional Fourier transform,
and optics people do not care about overall phases.

The Lie algebra of the group of integral transforms has as generators second-order differential operators

KBW, Canonical transforms I. Complex linear transforms, J. Math. Phys. **15**, 1295-1301 (1974)

The Lie algebra that generates LCTs:

$$\hat{J}\, f(\mathbf{x}) = -\mathrm{i}\frac{\partial}{\partial \tau}\int_{\mathsf{R}^N} \mathrm{d}\mathbf{x}'\, C_{\mathsf{M}(\tau)}(\mathbf{x},\mathbf{x}')\, f(\mathbf{x}')\Big|_{\tau=0},$$

Is a realization of sp(2,*R*) associated with paraxial optical elements

$$\begin{aligned}
\text{thin lens:} &\quad \exp\left(\mathrm{i}\tfrac{1}{2}\tau\hat{x}^2\right) = \mathcal{C}\begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix},\\
\text{free flight:} &\quad \exp\left(\mathrm{i}\tfrac{1}{2}\tau\hat{p}^2\right) = \mathcal{C}\begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix},\\
\text{magnifier:} &\quad \exp\left(\mathrm{i}\tfrac{1}{2}\tau(\hat{p}\hat{x}+\hat{x}\hat{p})\right) = \mathcal{C}\begin{pmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix},\\
\text{repulsive guide:} &\quad \exp\left(\mathrm{i}\tfrac{1}{2}\tau(\hat{p}^2-\hat{x}^2)\right) = \mathcal{C}\begin{pmatrix} \cosh\tau & -\sinh\tau \\ -\sinh\tau & \cosh\tau \end{pmatrix},\\
e^{\mathrm{i}\pi\tau/4}\times\text{Fourier}^{-\tau}: &\quad \exp\left(\mathrm{i}\tfrac{1}{2}\tau(\hat{p}^2+\hat{x}^2)\right) = \mathcal{C}\begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}.
\end{aligned}$$

There exist Bargmann-type Hilbert spaces of analytic functions where Complex LCT's are unitary.

$$\begin{pmatrix} \hat{z}^{\uparrow} \\ -\mathrm{i}\hat{z}^{\downarrow} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \hat{x}-\mathrm{i}\hat{p} \\ -\mathrm{i}(\hat{x}+\mathrm{i}\hat{p}) \end{pmatrix}, \quad -\mathrm{i}\equiv e^{-\mathrm{i}\pi/2}.$$

$$\mathrm{Re}\,(\mathrm{i}a/b) < 0. \quad -\pi < \arg b < 0.$$

Radial canonical transforms

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \qquad r \in \mathsf{R}_0^+ = [0,\infty), \quad \theta \in \mathsf{R} \bmod 2\pi.$$

Hilbert space on R+= [0, ∞)

$$(f,g)_{\mathcal{L}^2(\mathsf{R}^+)} := \int_0^\infty \mathrm{d}r\, f(r)^*\, g(r)$$

for m = 0, 1, 2, …

$$C_{\mathrm{M}}^{(m)}(r,r') = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\, C_{\mathrm{M}}(\mathbf{x},\mathbf{x}')e^{-\mathrm{i}m\theta}.$$

involves Bessel functions

$$= \frac{e^{\mathrm{i}\pi(m+1)/2}}{b}\exp\Big(\frac{\mathrm{i}}{2b}(dr^2+ar'^2)\Big)J_m\Big(\frac{rr'}{b}\Big),$$

M. Moshinsky, T.H. Seligman and KBW, Canonical radial transformations and the radial oscillator and Coulomb problems, *J. Math. Phys*. **13**, 901-907 (1972)

The **complex extensions** include Barut-Girardello-type integral transforms

KBW, Canonical transforms II. Complex radial transforms, *J. Math. Phys.* **15**, 2101-2111 (1974)

Hyperbolic canonical transforms

$$\left.\begin{array}{llll}\sigma=+: & x_1=\rho\cosh\zeta, & x_2=\rho\sinh\zeta, \\ \sigma=-: & x_1=\rho\sinh\zeta, & x_2=\rho\cosh\zeta,\end{array}\right\} \qquad \begin{array}{l}\rho,\,\zeta\in\mathsf{R},\\ \sigma:=\operatorname{sign}(x_1^2-x_2^2).\end{array}$$

Hilbert space on $R+ \oplus R+$

$$\mathbf{f}^{\varpi}(\rho)=\begin{pmatrix}f^{+,\varpi}(\rho)\\ f^{-,\varpi}(\rho)\end{pmatrix},\ \text{with}\ f^{\sigma,\varpi}(\rho)=\varpi f^{\sigma,\varpi}(-\rho),$$

$$(\mathbf{f},\mathbf{g})_{\mathcal{L}^2(\varpi,\mathsf{R}^+)}:=\sum_{\sigma\in\{+,-\}}\int_0^\infty d\rho\, f^{\sigma,\varpi}(\rho)^*\, g^{\sigma,\varpi}(\rho).$$

for $\sigma \in [0,\infty)$, $\varpi \in \pm$

$$\mathbf{f}_{\mathrm{M}}^{\varpi,s}(\rho)\equiv(\mathcal{C}_{\mathrm{M}}^{\varpi,s}\,\mathbf{f})(\rho)=\int_{\mathsf{R}+}d\rho'\,\mathbf{C}_{\mathrm{M}}^{(\varpi,s)}(\rho,\rho')\,\mathbf{f}(\rho'),$$

involves **Hankel** functions:

$$\mathbf{C}_{\mathrm{M}}^{(\varpi,s)}(\rho,\rho') = \begin{pmatrix} G_{\mathrm{M},+,+}(\rho,\rho')\,H_{+,+}^{(\varpi,s)}(\rho\rho'/b) & G_{\mathrm{M},+,-}(\rho,\rho')\,H_{+,-}^{(\varpi,s)}(\rho\rho'/b)\\ G_{\mathrm{M},-,+}(\rho,\rho')\,H_{-,+}^{(\varpi,s)}(\rho\rho'/b) & G_{\mathrm{M},-,-}(\rho,\rho')\,H_{-,-}^{(\varpi,s)}(\rho\rho'/b)\end{pmatrix}$$

$$G_{\mathrm{M},\sigma,\sigma'}(\rho,\rho') = \frac{\sqrt{\rho\rho'}}{2\pi\,|b|}\exp\Big(\mathrm{i}\frac{\sigma d\rho^2+\sigma' a\rho'^2}{2b}\Big),$$

$$H_{+,+}^{(\varpi,s)}(\xi) = \mathrm{i}\pi\big[\varpi e^{-\pi s}H_{2\mathrm{i}s}^{(1)}(\xi+\mathrm{i}0^+)-\varpi e^{\pi s}H_{2\mathrm{i}s}^{(2)}(\xi-\mathrm{i}0^+)\big] = \varpi H_{-,-}^{(\varpi,s)}(\xi),$$

KBW, Canonical Transforms IV. Hyperbolic transforms: continuous series of SL(2,R) representations, *J. Math. Phys*. **21**, 680-688 (1980)

no complex extension…

$$H_{+,-}^{(\varpi,s)}(\xi) = 4c_\xi^{\varpi,s}\,K_{2\mathrm{i}s}(|\xi|)=\varpi H_{-,+}^{(\varpi,s)}(\xi),$$

$$c_\xi^{+1,s}:=\cosh\pi s\ \text{ and }\ c_\xi^{-1,s}:=-\operatorname{sign}\xi\,\sinh\pi s.$$

What are canonical transforms?

Valentin Bargmann (1947), and Gel'fand and Naimark (1947 also) studied the unirreps of the 2+1 Lorentz algebra and group SO(2,1).

$$[\hat{J}_1, \hat{J}_2] = -\mathrm{i}\hat{J}_3, \quad [\hat{J}_2, \hat{J}_3] = \mathrm{i}\hat{J}_1, \quad [\hat{J}_3, \hat{J}_1] = \mathrm{i}\hat{J}_2,$$

We have a 2^{nd}-order diff op realization that transforms under $\mathrm{Sp}(2,R)$ as:

$$\begin{aligned} \hat{J}_1 &:= \frac{1}{4}\Big(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\gamma}{r^2} - r^2\Big), \\ \hat{J}_2 &:= \frac{-\mathrm{i}}{4}\Big(r\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\mathrm{d}}{\mathrm{d}r}r\Big), \\ \hat{J}_3 &:= \frac{1}{4}\Big(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\gamma}{r^2} + r^2\Big), \end{aligned}$$

$$\begin{pmatrix} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{pmatrix} \overset{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}{\longrightarrow} \begin{pmatrix} \frac{1}{2}(a^2-b^2-c^2+d^2) & bd-ac & \frac{1}{2}(a^2-b^2+c^2-d^2) \\ cd-ab & ad+bc & -cd-ab \\ \frac{1}{2}(a^2+b^2-c^2-d^2) & -bd-ac & \frac{1}{2}(a^2+b^2+c^2+d^2) \end{pmatrix} \begin{pmatrix} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{pmatrix}.$$

$$\begin{aligned} \hat{C} &:= \hat{J}_1^2 + \hat{J}_2^2 - \hat{J}_3^2 = (-\tfrac{1}{4}\gamma + \tfrac{3}{16})\, 1 \\ &=: k(1-k)\, 1, \qquad k : \text{Bargmann index.} \end{aligned}$$

$$\gamma = (2k-1)^2 - \tfrac{1}{4}, \qquad k = \tfrac{1}{2}(1 \pm \sqrt{\tfrac{1}{4}+\gamma}),$$

The centrifugal/centripetal potential divides unirreps into { **discrete**, **continuous**, **(and exceptional)** }

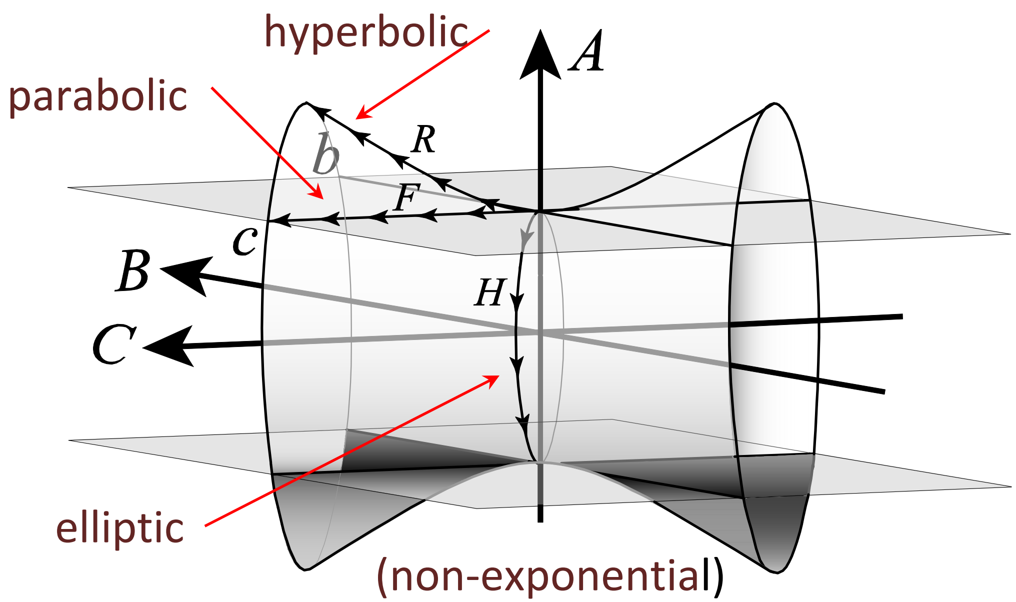
...and Sp(2,R) has 3 subgroup orbits

These provide the 'row' indices for matrix or integral kernel representations.

Elliptic orbit:
lower-bound or infinite **matrices**

Parabolic orbit
integral kernels of linear canonical transforms

Hyperbolic orbit
integral kernels



Thus we have
2 representation series in
3 subgroup reductions

Hilbert spaces undergoing unitary irreducible
Sp(2,R) Linear Canonical Transforms:

Representation series:	$\mathcal{D}_k^+$	$\mathcal{C}_s^\varepsilon$	
elliptic basis	$\ell^2(\mathsf{Z}_0^+)$	$\ell^2(\mathsf{Z})$	infinite or half-infinite discrete (vector) functions
parabolic basis	$\mathcal{L}^2(\mathsf{R}^+)$	$\mathcal{L}_2^2(\mathsf{R}^+)$	functions of a radius or pairs of the same.
hyperbolic basis	$\mathcal{L}^2(\mathsf{R})$	$\mathcal{L}_2^2(\mathsf{R})$	functions on the real line or pairs of the same.

$\mathcal{D}_k^+$ -elliptic   Acts on lower-bound infinite vectors

$$\mathbf{f}_{\mathrm{M}} \equiv \mathcal{C}_{\mathrm{M}} : \mathbf{f} = {}^0\mathbf{D}^k\,\mathbf{f},$$

$$f_{\mathrm{M};\,n} \equiv (\mathbf{f}_{\mathrm{M}})_n = \sum_{n'=0}^{\infty} {}^0\mathcal{D}^k_{k+n,k+n'}(\mathrm{M})\, f_{n'},$$

$$
\begin{aligned}
{}^0D^k_{m,m'}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &:= \left({}^0\Phi^k_m, \mathcal{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix} {}^0\Phi^k_{m'}\right) = \int_0^\infty \mathrm{d}r\, {}^0\Phi^k_m(r)^*\, \mathcal{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix} {}^0\Phi^k_{m'}(r) \\
&= \frac{2^{2k}\Gamma(m+m')}{\sqrt{\Gamma(k+m)\,\Gamma(1-k+m)\,\Gamma(k+m')\,\Gamma(1-k+m')}} \\
&\quad \times [(d-a)-\mathrm{i}(b+c)]^{m-k}[(a-d)-\mathrm{i}(b+c)]^{m'-k}[(a+d)+\mathrm{i}(b-c)]^{-m-m'} \\
&\quad \times {}_2F_1\left(\begin{matrix} k-m,\ k-m' \\ 1-m-m' \end{matrix} ; \frac{a^2+b^2+c^2+d^2+2}{a^2+b^2+c^2+d^2-2}\right).
\end{aligned}
$$

$\mathcal{D}_k^+$ -parabolic

Acts on functions on a radius

$$\mathbf{f}_{\mathrm{M}} \equiv \mathcal{C}_{\mathrm{M}} : \mathbf{f} = {}^{-}\mathbf{D}^k(\mathrm{M})\,\mathbf{f},$$

$$f_{\mathrm{M}}(r) \equiv (\mathbf{f}_{\mathrm{M}})(r) = \int_0^\infty \mathrm{d}r'\, {}^{-}\mathcal{D}^k_{r,r'}(\mathrm{M})\, f(r').$$

$${}^{-}\mathcal{D}^k_{\rho,\rho'}\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \left({}^{-}\Phi^k_{\rho}, \mathcal{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix} {}^{-}\Phi^k_{\rho'}\right) = \left({}^{-}\Phi^k_{\rho}, \mathcal{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathcal{C}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} {}^{+}\Phi^k_{\rho'}\right)$$

$$= \left({}^{-}\Phi^k_{\rho}, \mathcal{C}\begin{pmatrix} b & 0 \\ d & 1/b \end{pmatrix} \mathcal{C}\begin{pmatrix} 1 & -a/b \\ 0 & 1 \end{pmatrix} {}^{+}\Phi^k_{\rho'}\right)$$

$$= \frac{e^{-\mathrm{i}\pi k}}{b}\sqrt{\rho\rho'}\exp\left(\mathrm{i}\frac{d\rho^2 + a\rho'^2}{2b}\right) J_{2k-1}\left(\frac{\rho\rho'}{b}\right)$$

$$= \frac{2\,(\rho\rho')^{2k-1/2}}{(2\mathrm{i}b)^{2k}\,\Gamma(2k)} \exp\left(\mathrm{i}\frac{d\rho^2 - 2\rho\rho' + a\rho'^2}{2b}\right) {}_1F_1\left(\begin{matrix} 2k - \frac{1}{2} \\ 4k - 1 \end{matrix};\ \frac{2\mathrm{i}\rho\rho'}{b}\right).$$

LCTs on functions on the real line (Collins, Moshinsky & Quesne)

$$C_{\mathrm{M}}(x, x') = D_{\mathrm{M}}^{(1/4)}(r, r') + D_{\mathrm{M}}^{(3/4)}(r, r').$$

$\mathcal{D}_k^+$ -hyperbolic Acts on functions on the real line

$$\begin{aligned} \mathbf{f}_{\mathrm{M}} &\equiv \mathcal{C}_{\mathrm{M}} : \mathbf{f} = {}^2\mathbf{D}^k\,\mathbf{f}, \\ f_{\mathrm{M}}(\mu) &\equiv (\mathbf{f}_{\mathrm{M}})(\mu) = \int_{-\infty}^{\infty} \mathrm{d}\mu'\, {}^2D^k_{\mu,\mu'}(\mathrm{M})\, f(\mu'). \end{aligned}$$

$$\begin{aligned} {}^2D^k_{\mu,\mu'}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &:= \left({}^2\Phi^k_{\mu}, \mathcal{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix} {}^2\Phi^k_{\mu'}\right) \\ &= e^{-\mathrm{i}\pi k} 2^{\mathrm{i}(\mu'-\mu)} \frac{\Gamma(k-\mathrm{i}\mu)\,\Gamma(k+\mathrm{i}\mu')}{2\pi\,\Gamma(2k)} \\ &\quad \times b^{-2k}\left(\frac{-\mathrm{i}d}{b}\right)^{-k+\mathrm{i}\mu}\left(\frac{-\mathrm{i}a}{b}\right)^{-k-\mathrm{i}\mu'} {}_2F_1\left(\begin{matrix} k-\mathrm{i}\mu,\ k+\mathrm{i}\mu' \\ 2k \end{matrix}; \frac{1}{ad}\right). \end{aligned}$$

$$ {}^1D^k_{\mu,\mu'}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = {}^2D^k_{\mu,\mu'}\,\frac{1}{2}\begin{pmatrix} a+b+c+d & -a+b-c+d \\ -a-b+c+d & a-b-c+d \end{pmatrix}. $$

C_s^{ε} -elliptic

Acts on infinite vectors

$$\mathbf{f}_{\mathrm{M}} \equiv \mathcal{C}_{\mathrm{M}} : \mathbf{f} = {}^{0}\mathbf{C}^{\varepsilon,k}_{\mathrm{M}}\,\mathbf{f},$$

$$f_{\mathrm{M};\,m} \equiv (\mathbf{f}_{\mathrm{M}})_m = \sum_{m'=-\infty}^{\infty} {}^{0}C^{\varepsilon,k}_{m,m'}(\mathbf{M})\, f_{m'},$$

$${}^{0}C^{\varepsilon,k}_{m,m'}\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \left({}^{0}\Phi^{\varepsilon,k}_{m}, \mathcal{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix} {}^{0}\Phi^{\varepsilon,k}_{m'}\right) = \sum_{\sigma}\int_0^{\infty} \mathrm{d}r \cdots$$

$$\text{for } m \geq m', \quad = \quad \frac{2^{2m'}}{m'!}\sqrt{\frac{\Gamma(k{+}m)\,\Gamma(1{-}k{+}m)}{\Gamma(k{+}m)\,\Gamma(1{-}k{+}m)}}\frac{[(a{-}d){+}\mathrm{i}(b{+}c)]^{m-m'}}{[(a{+}d){+}\mathrm{i}(b{-}c)]^{m+m'}}$$

$$\times\, {}_2F_1\left(\begin{matrix} k{-}m',\; 1{-}k{-}m' \\ 1{+}m{-}m' \end{matrix}; -\tfrac{1}{4}(a^2{+}b^2{+}c^2{+}d^2{-}2)\right),$$

$$\text{for } m \leq m', \quad = \quad (-1)^{m'-m}\frac{2^{2m}}{m!}\sqrt{\frac{\Gamma(k{+}m')\,\Gamma(1{-}k{+}m')}{\Gamma(k{+}m)\,\Gamma(1{-}k{+}m)}}\frac{[(a{-}d){-}\mathrm{i}(b{+}c)]^{m'-m}}{[(a{+}d){+}\mathrm{i}(b{-}c)]^{m'+m}}$$

$$\times\, {}_2F_1\left(\begin{matrix} k{-}m,\; 1{-}k{-}m \\ 1{+}m'{-}m \end{matrix}; -\tfrac{1}{4}(a^2{+}b^2{+}c^2{+}d^2{-}2)\right).$$

$\mathcal{C}_s^\varepsilon$ -parabolic

Acts on two-functions on a radius

$$\mathbf{f}_{\mathrm{M}}(r) \equiv \left(\mathcal{C}_{\mathrm{M}} : \binom{f_{+1}}{f_{-1}}\right)(r) = \int_0^\infty \mathrm{d}r' \, {}^{-}\mathbf{C}_{\mathrm{M}}^{\varepsilon,k}(r,r')\,\mathbf{f}(r'),$$

$${}^{-}\mathbf{C}_{\mathrm{M}}^{\varepsilon,k}(r,r') = \begin{pmatrix} {}^{-}C^{\varepsilon,k}_{\mathrm{M};+1,+1}(r,r') & {}^{-}C^{\varepsilon,k}_{\mathrm{M};+1,-1}(r,r') \\ {}^{-}C^{\varepsilon,k}_{\mathrm{M};-1,+1}(r,r') & {}^{-}C^{\varepsilon,k}_{\mathrm{M};-1,-1}(r,r') \end{pmatrix},$$

$$({}^{-}\mathbf{C}_{\mathrm{M}}^{\varepsilon,k})_{\sigma,\sigma'}(r,r') = G_{\mathrm{M};\sigma,\sigma'}(r,r')\,H^{\varepsilon,k}_{\sigma,\sigma'}(-rr'/b),$$

$$G_{\mathrm{M};\sigma,\sigma'}(r,r') := \frac{\sqrt{rr'}}{2\pi\,|b|}\exp\left(\mathrm{i}\frac{d\sigma r^2 + a\sigma' r'^2}{2b}\right),$$

$$H^{\varepsilon,k}_{+1,+1}(\zeta) := \mathrm{i}\pi\left(e^{-\pi s} H^{(1)}_{2\mathrm{i}s}(\zeta+\mathrm{i}0^+) - h_\varepsilon e^{\pi s} H^{(2)}_{2\mathrm{i}s}(\zeta-\mathrm{i}0^+)\right)$$

$$= h_\varepsilon H^{\varepsilon,k}_{-1,-1}(\zeta) = h_\varepsilon H^{\varepsilon,k}_{+1,+1}(-\zeta) = H^{\varepsilon,1-k}_{+1,+1}(\zeta),$$

$$H^{\varepsilon,k}_{+1,-1}(\zeta) := 4(-\mathrm{sign}\,\zeta)\,g_\varepsilon(k)\,K_{2\mathrm{i}s}(|\zeta|)$$

$$= h_\varepsilon H^{\varepsilon,k}_{-1,+1}(\zeta) = h_\varepsilon H^{\varepsilon,k}_{+1,-1}(-\zeta) = h_\varepsilon H^{\varepsilon,1-k}_{+1,-1}(\zeta),$$

$$\varepsilon = 0: \quad h_\varepsilon = 1, \quad k-\tfrac{1}{2} = \mathrm{i}s, \quad s \geq 0,$$
$$\varepsilon = \tfrac{1}{2}: \quad h_\varepsilon = -1, \quad k-\tfrac{1}{2} = \mathrm{i}s, \quad s > 0,$$

C_s^ε -hyperbolic   Acts on two-functions on the real line

$$\mathbf{f}_{\mathrm{M}} \equiv \mathcal{C}_{\mathrm{M}} : \mathbf{f} = {}^2\mathbf{C}^{\varepsilon,k}\,\mathbf{f},$$

$$f_{\mathrm{M};\,\tau}(\mu) \equiv (\mathbf{f}_{\mathrm{M}})_\tau(\mu) = \sum_{\tau'\in\{-1,1\}} \int_{-\infty}^{\infty} \mathrm{d}\mu'\, {}^2C^{\varepsilon,k}_{\tau,\mu;\tau'\mu'}(\mathbf{M})\, f_{\tau'}(\mu')$$

$$\begin{aligned} {}^2C^{\varepsilon,k}_{\tau,\mu;\tau',\mu'}\begin{pmatrix} a & b \\ c & d\end{pmatrix} &:= \left({}^2\Phi^{\varepsilon,k}_{\tau,\mu}, \mathcal{C}\begin{pmatrix} a & b \\ c & d\end{pmatrix} {}^2\Phi^{\varepsilon,k}_{\tau',\mu'}\right) \\ &= \frac{(-\operatorname{sign} b)^{2\varepsilon} g_\varepsilon(k)}{2\pi}\Big[\Big(\alpha_k + \frac{\tau\tau' h_\varepsilon}{\alpha_k} + \tau'\beta_k + \frac{\tau h_\varepsilon}{\beta_k}\Big)T_k \\ &\qquad + \Big(h_\varepsilon \alpha_{1-k} + \frac{\tau\tau'}{\alpha_{1-k}} + \tau'\beta_{1-k} + \frac{\tau h_\varepsilon}{\beta_{1-k}}\Big)T_{1-k}\Big], \end{aligned}$$

where

$$\begin{aligned} T_k &:= \frac{\Gamma(1-2k)\,\Gamma(k-\mathrm{i}\mu)\,\Gamma(k+\mathrm{i}\mu')}{|a|^{k-\mathrm{i}\mu'}|b|^{\mathrm{i}(\mu-\mu')}|d|^{k-\mathrm{i}\mu}}\, {}_2F_1\left(\begin{matrix} k-\mathrm{i}\mu,\ k+\mathrm{i}\mu' \\ 2k\end{matrix};\ \frac{1}{ad}\right), \\ \alpha_k &:= \exp(\mathrm{i}\tfrac{1}{2}\pi[(k+\mathrm{i}\mu')\operatorname{sign} ab + (k-\mathrm{i}\mu)\operatorname{sign} bd]), \\ \beta_k &:= \exp(\mathrm{i}\tfrac{1}{2}\pi[-(k+\mathrm{i}\mu')\operatorname{sign} ab + (k-\mathrm{i}\mu)\operatorname{sign} bd]), \end{aligned}$$

Thus Sp(2,*R*) provides 6 faces of LCTs at least

$$f_{\mathrm{M}}^{k,\omega}(r) \;=\; \mathop{\mathsf{S}}_{r'(k,\omega)} D_{r,r'}^{k,\omega}(\mathrm{M})\, f(r'),$$

In the parabolic basis,

The C / M-Q LCTs belong to $D(+,1/4) + D(+,3/4)$.

Radial LCTs belong to the discrete series $D(+,k)$ for $k > 0$.

Hyperbolic LCTs belong to continuous series $C(\epsilon,s)$ for $k = 1/2 + \mathrm{i}\, s$.

In the elliptic basis

we can have lower-bound infinite matrix LCTs in the discrete series,

or infinite matrix LCTs in the continuous series.

The hyperbolic basis for 2-component functions remains unused.

The exceptional interval $0 < k < 1$ has 1-param self-adjoint extensions.

The exceptional representation series is *terra incognita* (*hic vivunt leones!*).

Conserved uncertainty relations

$$\Delta_\psi(J_-)\,\Delta_\psi(J_+) = \Delta_\psi(J_-)\,\Delta_{\tilde\psi}(J_-) \geq \tfrac{1}{4}|(\psi, J_2\psi)|^2,$$

$$\langle r^2\rangle_\psi\, \langle p^2\rangle_\psi \geq \langle rp\rangle_\psi^2$$

for `classical’ LCTs

There has been much recent interest in FINITE LCTs to permit procesing by computers

Preferably using the FFT algorithm for $F_{m,m'} = \frac{1}{\sqrt{N}} \exp\left(-\,\mathrm{i}\frac{2\pi\, m\, m'}{N}\right),$

So, people use: $(\mathcal{C}_{\mathrm{M}}\,\mathbf{f})_m \;:=\; \sum_{m'=1}^{N} \frac{1}{\sqrt{N}} \exp\left(\frac{\mathrm{i}}{2b}(dx_m^2 - 2x_m x_{m'} + a x_{m'}^2)\right) f_{m'},$

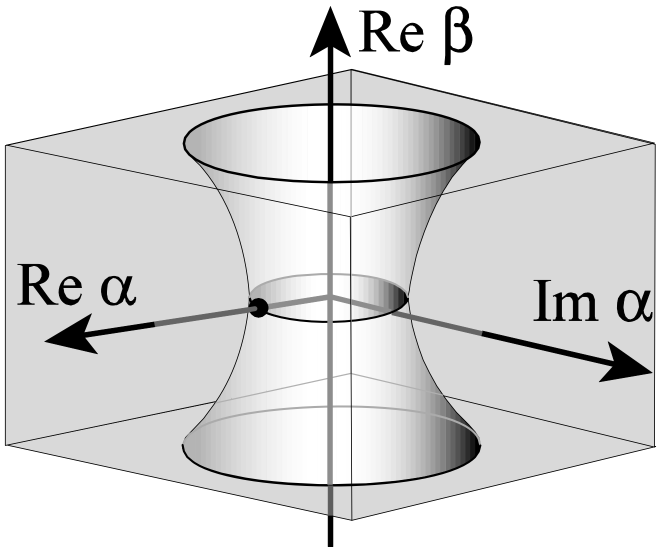
But of course… non-compact groups do not have unitary finite-dimensional faithful representations !!!

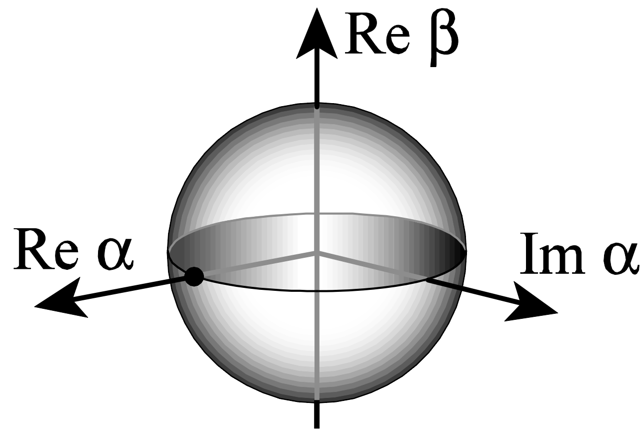
A 'finite fractional Fourier transform' presents multi-solutions, and a toroidal phase space will not rotate continuously.

$$\mathbf{F}^{\nu}\mathbf{v}^{(\varphi_n,j)} = \exp[-\mathrm{i}\tfrac{1}{2}\pi(4j+n)\nu]\,\mathbf{v}^{(\varphi_n,j)} = \varphi_n^{\nu}\exp(-2\pi\mathrm{i}j\nu)\,\mathbf{v}^{(\varphi_n,j)},$$

$$\mathbf{F}_{\mathrm{V}}^{\nu} = \sum_{n=0}^{3} \varphi_n^{\nu}\mathbf{V}^{(\varphi_n)}\mathbf{\Phi}^{(\varphi_n)}(\nu)\,\mathbf{V}^{(\varphi_n)\dagger},$$

Finite data sets divide us into two cities:
in one, those who believe in *symmetries*,
in the other, those who want *fast results*.





Use discretized version of LCTs
not unitary, do not compose
toroidal phase space a mess...

Use the group **SU(2)**.
Enjoy unitarity, composition,
and a nice spherical phase space.

¡ Muchas gracias!

PAPIIT-UNAM
and SEP-CONACYT
projects
Óptica Matemática

¡ Muchas gracias!

With best admiration and wishes
for Prof. Francesco Iachello
on his 70 years of success !

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