Beautiful Graphene, Photonic Crystals, Schrödinger and Dirac Billiards and Their Spectral Properties

• Something about graphene and microwave billiards

• Dirac spectrum in a photonic crystal
  • Experimental setup
  • Transmission and reflection spectra

• Photonic crystal in a box: Dirac billiards
  • Measured spectra
  • Density of states
  • Spectral properties

• Outlook

Supported by DFG within SFB 634
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Nobel Prize in Physics 2010

Andre Geim
Konstantin Novoselov

The Nobel Prize in Physics 2010 was awarded jointly to Andre Geim and Konstantin Novoselov "for groundbreaking experiments regarding the two-dimensional material graphene"
Graphene

- “What makes graphene so attractive for research is that the spectrum closely resembles the Dirac spectrum for massless fermions.”
  M. Katsnelson, Materials Today, 2007

- Two triangular sublattices of carbon atoms
- Near each corner of the first hexagonal Brillouin zone the electron energy $E$ has a conical dependence on the quasimomentum
  - $E = \hbar v_F k$ but low $v_F \approx c/300$
- Experimental realization of graphene in analog experiments of microwave photonic crystals
Quantum Billiards and Microwave Billiards

Quantum billiard

\[ \left( \frac{\hbar}{2m} \Delta + E \right) \Psi = 0, \quad \Psi|_{\partial \Omega} = 0 \]

Microwave billiard

\[ (\Delta + k^2) E_z = 0, \quad E_z|_{\partial \Omega} = 0 \]

Analogy for \( \lambda > 2d \)

- eigenvalue \( E \) ↔ wave number \( k = \frac{2\pi f}{c} \)
- eigenfunction \( \Psi \) ↔ electric field strength \( E_z \)
Measurement Principle

- Measurement of scattering matrix element $S_{21}$

$$\frac{P_{out,2}}{P_{in,1}} = |S_{21}|^2$$

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Resonance spectrum

$\rho(f) = \sum \delta(f - f_v)$

Length spectrum (Gutzwiller)

$\tilde{\rho}(l)$

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rf power rf power

in

out

rf power rf power

200 mm

560 mm
Open Flat Microwave Billiard: Photonic Crystal

• A photonic crystal is a structure, whose electromagnetic properties vary periodically in space, e.g. an array of metallic cylinders
  → open microwave resonator

• Flat “crystal” (resonator) → E-field is perpendicular to the plates (TM₀ mode)
• Propagating modes are solutions of the scalar Helmholtz equation
  → Schrödinger equation for a quantum multiple-scattering problem
  → Numerical solution yields the band structure
Dispersion relation $\omega(\mathbf{k})$ of a photonic crystal exhibits a band structure analogous to the electronic band structure in a solid.

The triangular photonic crystal possesses a conical dispersion relation → Dirac spectrum with a Dirac point where bands touch each other.

The voids form a honeycomb lattice like atoms in graphene.
Effective Hamiltonian around the Dirac Point

• Close to Dirac point the effective Hamiltonian is a 2x2 matrix

\[ \hat{H}_{\text{eff}} = \omega_D \mathbb{1} + v_D (\delta k_x \hat{\sigma}_x + \delta k_y \hat{\sigma}_y) \]

• Substitution \( \delta k_x \rightarrow -i \partial_x \) and \( \delta k_y \rightarrow -i \partial_y \) leads to the Dirac equation

\[
\begin{pmatrix}
0 & \partial_x - i \partial_y \\
\partial_x + i \partial_y & 0
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
= i \frac{\omega - \omega_D}{v_D}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\]

• Experimental observation of a Dirac spectrum in open photonic crystal

(S. Bittner et al., PRB 82, 014301 (2010))

• Scattering experiment
Scattering Experiment

- Horn antenna emits approximately plane waves
- VNA measures the modulus of the scattering matrix given by
  \[ |S_{ba}|^2 = \frac{P_b}{P_a} \]
- Transmission: \(|S_{ab}|^2, |S_{ba}|^2\)
- Reflection: \(|S_{aa}|^2, |S_{bb}|^2\)
Experimental Realization of 2D Photonic Crystal

- # cylinders: \( 23 \times 38 = 874 \)
- Cylinder radius: \( R = 5 \text{ mm} \)
- Lattice constant: \( a = 20 \text{ mm} \)
- Crystal size: \( 400 \times 900 \times 8 \text{ mm} \)
- Frequency: \( f_{\text{max}} = 19 \text{ GHz} \)

First step: experimental observation of the band structure
Transmission through the Photonic Crystal

- Transmission spectrum possesses two stop bands
- Comparison with calculated band structure
Projected Band Diagram

- The density plot of the 1st frequency band
- The projected band diagram along the irreducible Brillouin zone ΓMK
- The 1st and 2nd frequency bands touch each other at the corners of the Brillouine zone → Dirac Point
Transmission through the Photonic Crystal

- The positions of measured stop bands coincide with the calculated ones → lattice parameters chosen correctly
- Dirac point is not sufficiently pronounced in the transmission spectra → single antenna reflection measurement
Single Antenna Reflection Spectrum

• Measurement with a wire antenna $a$ put through a drilling in the top plate $\rightarrow$ point like field probe

• Characteristic cusp structure around the Dirac frequency
• Van Hove singularities at the band saddle point $|\nabla_{\vec{k}} \omega(\vec{k})| = 0$
• Next: analysis of the measured spectrum
Local Density of States and Reflection Spectrum

• The scattering matrix formalism relates the reflection spectra to the local density of states (LDOS)

\[1 - |S_{aa}(f)|^2 \propto L(\vec{r}_a, f)\]

• LDOS

\[L(\vec{r}, f) \propto \int_{BZ} |\psi(\vec{k}, \vec{r})|^2 \frac{1}{2\pi} \delta(f - f(\vec{k})) d^2k\]

• LDOS around the Dirac point (Wallace, 1947)

\[L(\vec{r}_a, f) \sim \frac{\langle |\psi(\vec{r}_a)|^2 \rangle}{v_D^2} |f - f_D|\]

• Three parameter fit formula

\[|S_{aa}(f)|^2 = D - C |f - f_D|\]

fit parameters
Reflection Spectra

- Description of experimental reflection spectra $|S_{aa}(f)|^2 = D - C |f - f_D|$

antenna $a$ in the middle of the crystal

$|S_{aa}|^2$

\[ f_D = 13.797 \pm 0.004 \text{ GHz} \]

antenna $a$ 6 rows apart from the boundary

$|S_{aa}|^2$

\[ f_D = 13.793 \pm 0.002 \text{ GHz} \]

- Experimental Dirac frequencies agree with calculated one, $f_D = 13.81 \text{ GHz}$, within the standard error of the fit
- Oscillations around the mean intensity $\rightarrow$ origin?
Comparison with STM Measurements

- Tunneling conductance is proportional to LDOS
- Similarity with measured reflection spectrum of the photonic crystal
- Oscillations in STM are not as pronounced due to the large sample size
- Finestructure in the photonic crystal shows fluctuations (RMT)
Summary I

• Connection between reflection spectra and LDOS is established

• Cusp structure in the reflection spectra is identified with the Dirac point

• Photonic crystal simulates one particle properties of graphene

• Results are published in Phys. Rev. B 82 014301 (2010)

• Measured also transmission near the Dirac Point

• Dirac billiards
Dirac Billiard

• Photonic crystal → box: bounded area = billiard

• 888 cylinders (scatterers) milled out of a brass plate
• Height $d = 3\, \text{mm} \rightarrow f_{\text{max}}^{2D} = 50\, \text{GHz}$ for 2D system
• Lead plated → superconducting below 7.2 K → high Q value
• Boundary does not violate the translation symmetry → no edge states
• Relativistic massless spin-one half particles in a billiard (Berry and Mondragon, 1987)
Transmission Spectrum at 4 K

- Pronounced stop bands
- Quality factors $> 5 \cdot 10^5$
- $\langle \Gamma \rangle / \langle D \rangle = 10^{-3} \rightarrow$ complete spectrum
- Altogether 5000 resonances observed
Density of States of the Measured Spectrum and the Band Structure

- Positions of stop bands are in agreement with calculation
- DOS related to slope of a band
- Dips correspond to Dirac points
- High DOS at van Hove singularities → ESQPT?
- Flat band has very high DOS
- Qualitatively in good agreement with prediction for graphene
  (Castro Neto et al., RMP 81,109 (2009))
Integrated Density of States: 1\textsuperscript{st} and 2\textsuperscript{nd} Bands

- Does not follow Weyl law for 2D resonators ($N_{Weyl}(f) = \frac{4\pi A}{c^2} f^2$)
- Small slope at the Dirac frequency $\rightarrow$ DOS nearly zero
- Nearly symmetric with respect to the Dirac frequency
- Two parabolic branches
Integrated DOS near Dirac Point

- Weyl law for Dirac billiard $N(k) = \frac{A}{2\pi} k^2 + \frac{U_{zz}}{\pi} k + C$ (J. Wurm et al., PRB 84, 075468 (2011))
  - $U_{zz}$ is length of zigzag edges
  - $k = 2\pi \frac{|f - f_D|}{v_D}$
  - group velocity $v_D$ is a free parameter

- Same area $A$ for two branches, but different group velocities $\rightarrow \alpha \neq \beta$
  $\rightarrow$ electron-hole asymmetry like in graphene
Spectral Properties of a Rectangular Dirac Billiard: Nearest Neighbour Spacing Distribution

- 159 levels around Dirac point
- Rescaled resonance frequencies such that $\langle s_i \rangle = 1$
- Poisson statistics
- Similar behavior at second Dirac point

$s_i = f_{i+1} - f_i$

![Graph showing the distribution of $s_i$ with bars and a fitted line.](image)
NND: 3\textsuperscript{rd} Band

- Very dense spectrum $\langle \Gamma \rangle / \langle D \rangle \approx 10^{-1}$
- Unfolding with a polynom of 8\textsuperscript{th} order
- Missing levels?
- Seems to agree with GOE
Summary II

- Photonic crystal simulates one particle properties of graphene

- Observation of edge states in the Dirac billiard (not shown)

- Realisation of superconducting microwave Dirac billiard i.e. photonic crystal in a metallic box serves as a model for a relativistic quantum billiard

- Experimental DOS agrees with calculated photonic band structure

- Fluctuation properties of the spectrum were investigated

- Open problems:
  (i) Length spectrum of periodic orbits
  (ii) Do we see an excited state quantum phase transition?
Excited State Quantum Phase Transitions

- Francesco Iachello at the “6th International Workshop in Shape-Phase Transitions and Critical-Point Phenomena in Nuclei” Darmstadt 2012

- Control parameter $\xi$
- At the separatrix the density of states diverges (Caprio, Cejnar, Iachello, 2008)
Excited State Quantum Phase Transition in Dirac Billiard

- Experimental density of states in the Dirac billiard

- Van Hove singularities at saddle point: density of states diverges at $k=M$

- Possible control parameters
  - chemical potential
  - sublattice dependent potential
  - …
The mode has now several deformed nuclei, with closer to those predicted in RPA the subject of several experiments. The purpose of this work has clearly been much stimulated by Franco Iachello, we thank him and Wolfgang Knüpfer and Bernard Metsch for several discussions.

638 citations

All the best and many more fruitful years together!