Advances in the Calculation of Double Beta Decay

Jenni Kotila

Center for Theoretical Physics, Sloane Physics Laboratory, Yale University,
New Haven, Connecticut, 06520-8120, USA

Yale

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on the occasion of his 70th birthday

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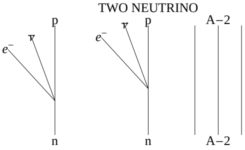
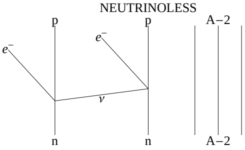
- Nucleus (A, Z) decays to nucleus $(A, Z \pm 2)$ by emitting two electrons (or positrons) + other light particles

- Modes of interest to this talk: $2\nu\beta^-\beta^-$ mode: $(A, Z) \rightarrow (A, Z + 2) + 2e^- + 2\bar{\nu}$
  - Allowed by the standard model



- and the process $0\nu\beta^-\beta^-$: $(A, Z) \rightarrow (A, Z + 2) + 2e^-$
  - Decay probability proportional to the square of the average light neutrino mass $\langle m_\nu \rangle$, or in case of heavy neutrinos to the lepton nonconserving parameter $|\eta|$



Introduction

- For processes allowed by the standard model the half-life is

$$\left[\tau_{1/2}^{2\nu}\right]^{-1} = G_{2\nu} g_A^4 |m_e c^2 M^{(2\nu)}|^2$$

- and for neutrinoless modes

$$\left[\tau_{1/2}^{0\nu}\right]^{-1} = G_{0\nu} g_A^4 |M^{(0\nu)}|^2 |f(m_i, U_{ei})|^2$$

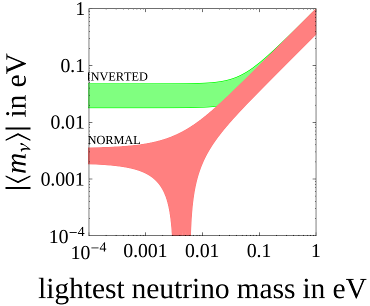
- $G_{2\nu}$ and $G_{0\nu}$ are the phase space factors
- g_A is the axial vector coupling constant (effective value essentially model dependent!)
- $M^{(2\nu)}$ and $M^{(0\nu)}$ are the nuclear matrix elements
- $f(m_i, U_{ei})$ contains the physics beyond standard model

For both processes, two crucial ingredients are the phase space factors and the nuclear matrix elements!

- Light neutrinos:

$$f(m_i, U_{ei}) = \frac{\langle m_\nu \rangle}{m_e} = \frac{1}{m_e} \sum_{k=light} (U_{ek})^2 m_k$$

- Advance: The average light neutrino mass is now well constrained by atmospheric, solar, reactor and accelerator neutrino oscillation experiments



- Heavy neutrinos:

$$f(m_i, U_{ei}) = |\eta| = m_p \left\langle m_{\nu_h}^{-1} \right\rangle = m_p \sum_{k=heavy} (U_{ek_h})^2 \frac{1}{m_{k_h}}$$

Calculation of PSF: Electron wave functions

- The key ingredient for the evaluation of phase space factors (and thus double beta decay) are the electron wave functions
- We use positive energy Dirac central field wave functions

$$\psi_{\epsilon\kappa\mu}(\mathbf{r}) = \begin{pmatrix} g_\kappa(\epsilon, r)\chi_\kappa^\mu \\ if_\kappa(\epsilon, r)\chi_{-\kappa}^\mu, \end{pmatrix}$$

- χ_κ^μ = spherical spinors, $g_\kappa(\epsilon, r)$ and $f_\kappa(\epsilon, r)$ = radial functions
- $g_\kappa(\epsilon, r)$ and $f_\kappa(\epsilon, r)$ satisfy the radial Dirac equations:

$$\begin{cases} \dfrac{dg_\kappa(\epsilon, r)}{dr} = -\dfrac{\kappa}{r} g_\kappa(\epsilon, r) + \dfrac{\epsilon - V + m_e c^2}{c\hbar} f_\kappa(\epsilon, r) \\ \dfrac{df_\kappa(\epsilon, r)}{dr} = -\dfrac{\epsilon - V - m_e c^2}{c\hbar} g_\kappa(\epsilon, r) + \dfrac{\kappa}{r} f_\kappa(\epsilon, r) \end{cases}$$

Calculation of PSF: Electron wave functions

- General analytical solution does not exist for screened potential BUT
- numerical solution by Salvat *et al.*, Comput. Phys. Commun. 90, 151 (1995)
- Radial equations solved using piecewise exact power series expansion of the radial functions
- Radial wave functions are normalized so that they asymptotically oscillate with

$$\begin{pmatrix} g_\kappa(\epsilon, r) \\ f_\kappa(\epsilon, r) \end{pmatrix} \sim e^{-i\delta_\kappa} \frac{\hbar}{pr} \begin{pmatrix} \sqrt{\frac{\epsilon + m_e c^2}{2\epsilon}} \sin(kr - l\frac{\pi}{2} - \eta \ln(2kr) + \delta_\kappa) \\ \sqrt{\frac{\epsilon - m_e c^2}{2\epsilon}} \cos(kr - l\frac{\pi}{2} - \eta \ln(2kr) + \delta_\kappa) \end{pmatrix}$$

- $\epsilon = \sqrt{(m_e c^2)^2 + (pc)^2}$
- $k \equiv \frac{p}{\hbar} = \frac{\sqrt{\epsilon^2 + (m_e c^2)^2}}{c\hbar}$ is the electron wave number
- $\eta = Ze^2/\hbar v$ is the Sommerfeld parameter
- δ_k is the phase shift

Calculation of PSF: Electron wave functions

- Electron scattering wave function can be expanded in terms of spherical waves

$$e_s(\epsilon, \mathrm{r}) = e_s^{S_{1/2}}(\epsilon, \mathrm{r}) + e_s^{P_{1/2}}(\epsilon, \mathrm{r}) + e_s^{P_{3/2}}(\epsilon, \mathrm{r}) + \ldots$$

where

$$e_s^{S_{1/2}}(\epsilon, \mathrm{r}) = \begin{pmatrix} g_{-1}(\epsilon, r)\chi_s \\ f_1(\epsilon, r)(\hat{\mathrm{p}} \cdot \vec{\sigma})\chi_s \end{pmatrix}$$

$$e_s^{P_{1/2}}(\epsilon, \mathrm{r}) = \begin{pmatrix} ig_1(\epsilon, r)(\hat{\mathrm{r}} \cdot \vec{\sigma})(\hat{\mathrm{p}} \cdot \vec{\sigma})\chi_s \\ -if_{-1}(\epsilon, r)(\hat{\mathrm{r}} \cdot \vec{\sigma})\chi_s \end{pmatrix}$$

$$e_s^{P_{3/2}}(\epsilon, \mathrm{r}) = \begin{pmatrix} ig_{-2}(\epsilon, r)[3(\hat{\mathrm{r}} \cdot \hat{\mathrm{p}}) - (\hat{\mathrm{r}} \cdot \vec{\sigma})(\hat{\mathrm{p}} \cdot \vec{\sigma})]\chi_s \\ if_2(\epsilon, r)[3(\hat{\mathrm{r}} \cdot \hat{\mathrm{p}})(\hat{\mathrm{p}} \cdot \vec{\sigma}) - (\hat{\mathrm{r}} \cdot \vec{\sigma})]\chi_s \end{pmatrix}$$

Calculation of PSF: Potential

- To simulate realistic situation, we take into account the finite nuclear size and the electron screening
- Thomas-Fermi screening function, $\varphi(r) = Z_{eff}/Z_d$, obtained by Majorana solution of Thomas Fermi equation (Esposito, Am. J. Phys. 70, 852 (2002))

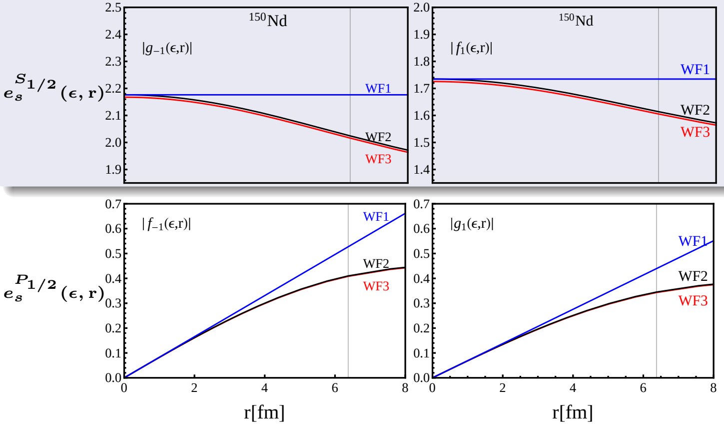
$$\frac{d^2\varphi}{dx^2} = \frac{\varphi^{3/2}}{\sqrt{x}}$$

  - $x = r/b$ with $b = \frac{1}{2}\left(\frac{3\pi}{4}\right)^{2/3} \frac{\hbar^2}{m_e e^2} Z_d^{-1/3} \simeq 0.8853 a_0 Z_d^{-1/3}$
  - boundary conditions: $\varphi(0) = 1$, $\varphi(\infty) = \frac{2}{Z_d}$
- The screened potential is then written as

$$V(r) = \varphi(r) \times \begin{bmatrix} \frac{-Z_d(\alpha\hbar c)}{r}, & r \geq R \\ -Z_d(\alpha\hbar c)\left(\frac{3-(r/R)^2}{2R}\right), & r < R \end{bmatrix}$$

Calculation of PSF: Radial wave functions

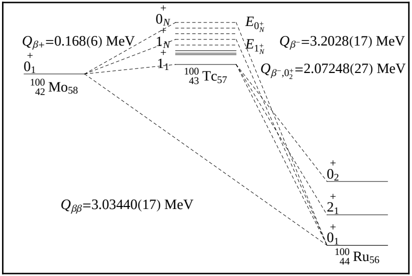
Example of obtained radial wave functions: ^{150}Nd decay,
$Z_d = 62$ at $\epsilon = 2.0$MeV, $R(150) = 6.38$fm

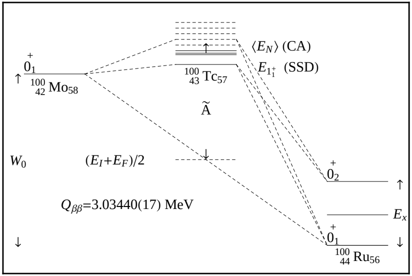


WF1 = Leading finite size Coulomb (previous studies)
WF2 = Exact finite size Coulomb
WF3 = Exact finite size Coulomb & electron screening

$2\nu\beta^-\beta^-$

- Example decay scheme for $2\nu\beta^-\beta^-$: ^{100}Mo $\rightarrow ^{100}$Ru





- Possible decays: $0^+ \rightarrow 0^+$ and $0^+ \rightarrow 2^+$
- $Q_{\beta\beta}$ is the reaction Q-value
- Separation of PSF and NME can be done for:
  - Closure Approximation (CA)
    $$\tilde{A} = \frac{1}{2}W_0 + \langle E_N \rangle - E_I = \frac{1}{2}(Q_{\beta\beta} + 2m_e c^2) + \langle E_N \rangle - E_I \simeq 1.12A^{1/2}$$
  - Single State Dominance hypothesis:
    $$\tilde{A} = \frac{1}{2}W_0 + E_{1^+_1} = \frac{1}{2}(Q_{\beta\beta} + 2m_e c^2) + E_{1^+_1} - E_I$$

$2\nu\beta^-\beta^-$: PSF

- Differential rate for $0^+ \to 0^+$ $2\nu\beta^-\beta^-$-decay is given by

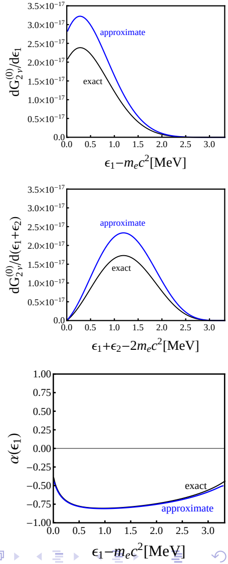
$$dW_{2\nu} = \left(a^{(0)} + a^{(1)} \cos\theta_{12}\right) w_{2\nu} d\omega_1 d\epsilon_1 d\epsilon_2 d(\cos\theta_{12})$$

  - $w_{2\nu} = \frac{(G\cos\theta_C)^4}{64\pi^7\hbar}\omega_1^2\omega_2^2(p_1c)(p_2c)\epsilon_1\epsilon_2$
  - $a^{(0)}$ and $a^{(1)}$ are functions of $\langle K_N\rangle$ and $\langle L_N\rangle$ (functions of $\epsilon_1, \epsilon_2, \omega_1, \omega_2, \tilde{A}, Q_{\beta\beta}$), $|M^{(2\nu)}|$ and $f_{11}^{(0)}$ and $f_{11}^{(1)}$ (products of $g_{-1}(\epsilon, R)$ and $f_1(\epsilon, R)$)
- All quantities of interest are obtained by integration of $dW_{2\nu}$
- All quantities are separated into a phase space factor (independent of NMEs) and NMEs
- Phase space factor of interest

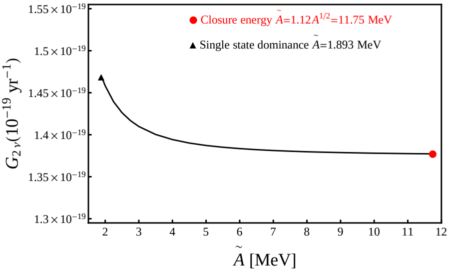
$$G_{2\nu} = \frac{1}{(m_ec^2)^2}\frac{2\tilde{A}^2}{3\ln 2}\int_{m_ec^2}^{Q_{\beta\beta}+m_ec^2}\int_{m_ec^2}^{Q_{\beta\beta}+m_ec^2-\epsilon_1}\int_0^{Q_{\beta\beta}-\epsilon_1-\epsilon_2} f_{11}^{(0)} \times \left(\langle K_N\rangle^2 + \langle L_N\rangle^2 + \langle K_N\rangle\langle L_N\rangle\right) w_{2\nu} d\omega_1 d\epsilon_2 d\epsilon_1$$

From these, we obtain:

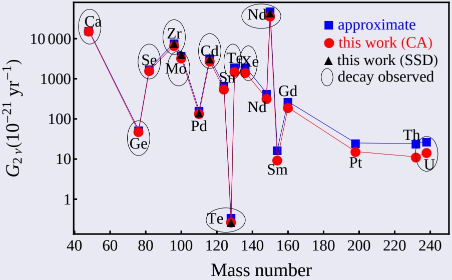
- The half-life
  $$\left[\tau_{1/2}^{2\nu}\right]^{-1} = G_{2\nu} g_A^4 \left| m_e c^2 M^{(2\nu)} \right|^2$$
- The differential decay rate
  $$\frac{dW_{2\nu}}{d\epsilon_1} = \mathcal{N}_{2\nu} \frac{dG_{2\nu}}{d\epsilon_1}$$
  where $\mathcal{N}_{2\nu} = g_A^4 \left| m_e c^2 M^{(2\nu)} \right|^2$.
- The summed energy spectrum of the two electrons
  $$\frac{dW_{2\nu}}{d(\epsilon_1+\epsilon_2)} = \mathcal{N}_{2\nu} \frac{dG_{2\nu}}{d(\epsilon_1+\epsilon_2)}$$
- The angular correlation between the two electrons
  $$\alpha(\epsilon_1) = \frac{dG_{2\nu}^{(1)}/d\epsilon_1}{dG_{2\nu}/d\epsilon_1}$$
  where $G_{2\nu}^{(1)}$ is also obtained from $dW_{2\nu}$ ($a^{(1)}$ term)



$2\nu\beta^-\beta^-$: Results



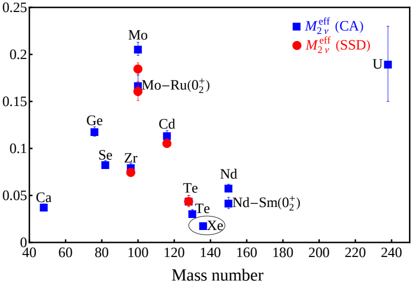
- $G_{2\nu}$ as a function of $\tilde{A}$
  - Stays almost constant, which means CA good approximation
  - EXCEPT near the threshold $\langle E_N \rangle$, which case SSD good approximation



- Current $2\nu\beta^-\beta^-$ phase space factors (red: CA and black: SSD) compared to previous calculations (blue)
- Marked nuclei: Decay observed

$2\nu\beta^-\beta^-$: $M_{2\nu}^{eff}$ from experiments

Nucleus	$\tau_{1/2}^{2\nu}$ (10^{18} y) exp*
^{48}Ca	44^{+6}_{-5}
^{76}Ge	1500 ± 100
^{82}Se	92 ± 7
^{96}Zr	23 ± 2
^{100}Mo	7.1 ± 0.4
^{100}Mo-^{100}Ru(0_2^+)	590^{+80}_{-60}
^{116}Cd	28 ± 2
^{128}Te	1900000 ± 400000
^{130}Te	680^{+120}_{-110}
^{136}Xe	2110 ± 250
^{150}Nd	8.2 ± 0.9
^{150}Nd-^{150}Sm(0_2^+)	133^{+45}_{-26}
^{238}U	2000 ± 600



- $|M_{2\nu}^{eff}|^2$ is obtained from the measured half-life by

$$|M_{2\nu}^{eff}|^2 = \left[\tau_{1/2}^{2\nu} \times G_{2\nu}\right]^{-1}$$

- Extracted dimensionless quantity

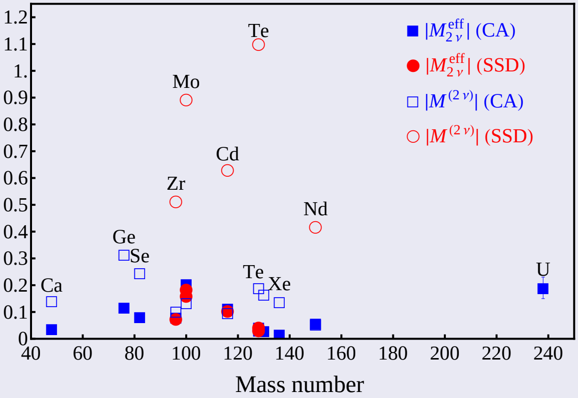
$$|M_{2\nu}^{eff}|^2 = g_A^4 |(m_e c^2) M^{(2\nu)}|^2$$

* A.S. Barabash, Phys. Rev. C **81**, 035501 (2010).

Smallest $M_{2\nu}^{eff}$ for ^{136}Xe, just recently measured!

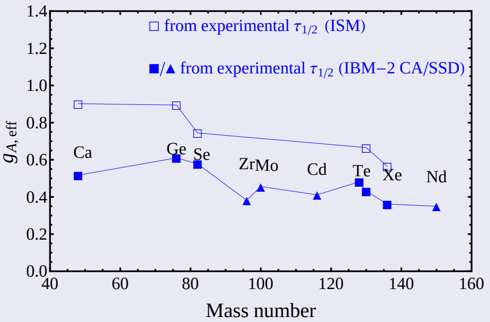
$2\nu\beta^-\beta^-$: $M_{2\nu}^{eff}$ from experiments

- Now, if we add to the same figure the theoretical IBM-2 matrix elements $|M^{(2\nu)}| = |\frac{M_{GT}^{(2\nu)}}{\tilde{A}_{GT}} - \left(\frac{g_V}{g_A}\right)^2 \frac{M_F^{(2\nu)}}{\tilde{A}_F}|$ which DO NOT include the factor g_A^2...
- ... but they are still much larger than $M_{2\nu}^{eff}$
- $g_{A,eff} < 1.0$, at least in the case of $2\nu\beta^-\beta^-$!



$2\nu\beta^-\beta^-$: Determination of effective g_A

- g_A is renormalized in nuclei
- renormalization depends on the size of the model space
  - IBM-2: small model space
  - ISM: large model space
- $g_{A,eff}$ can be extracted comparing $|M_{2\nu}^{eff}|$ and $|M_{2\nu}|$



* ISM NMEs from E. Caurier *et al.*, Int. J. Mod. Phys. E **16**, 552 (2007).

$2\nu\beta^-\beta^-$: Determination of effective g_A

- g_A is renormalized in nuclei
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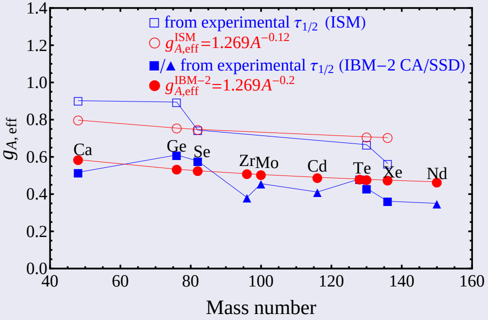
  - IBM-2: small model space
  - ISM: large model space
- $g_{A,eff}$ can be extracted comparing $|M_{2\nu}^{eff}|$ and $|M_{2\nu}|$
- Assumption: $g_{A,eff}$ is a smooth function of A



- Parametrization:
  $g_{A,eff} = 1.269A^{-\gamma}$
  - IBM-2: $\gamma = 0.2$
  - ISM: $\gamma = 0.12$

* ISM NMEs from E. Caurier *et al.*, Int. J. Mod. Phys. E **16**, 552 (2007).

$0\nu\beta^-\beta^-$: PSF

- The differential rate $dW_{0\nu}$ for the decay is essentially a function of electron energies and radial wave functions
- By integration of $dW_{0\nu}$ we get

$$G_{0\nu} = \frac{1}{g_A^4 (4R^2)} \frac{2}{\ln 2} \int_{m_e c^2}^{Q_{\beta\beta}+m_e c^2} f_{11}^{(1)} w_{0\nu} d\epsilon_1, \qquad R = r_0 A^{1/3} = 1.2 A^{1/3},$$

where $w_{0\nu} = \frac{g_A^4 (G\cos\theta_C)^4}{16\pi^5} (m_e c^2)^2 (\hbar c^2)(p_1 c)(p_2 c)\epsilon_1 \epsilon_2$

- The half-life is then

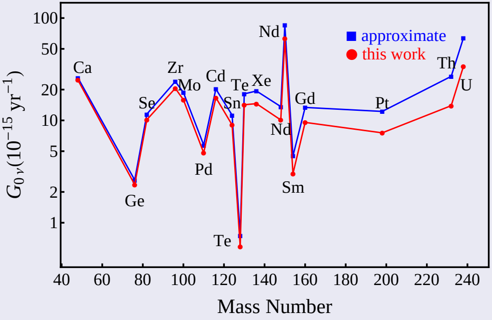
$$\left[\tau_{1/2}^{0\nu}\right]^{-1} = G_{0\nu} g_A^4 \left(\frac{\langle m_\nu \rangle}{m_e}\right)^2 |M_{GT}^{(0\nu)} - \left(\frac{g_V}{g_A}\right)^2 M_F^{(0\nu)} + M_T^{(0\nu)}|^2$$

- The single electron spectrum

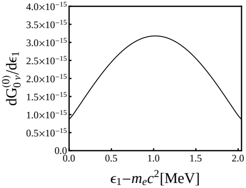
$$\frac{dW_{0\nu}}{d\epsilon_1} = \mathcal{N}_{0\nu} \frac{dG_{0\nu}}{d\epsilon_1} = \mathcal{N}_{0\nu} \left[2 f_{11}^{(0)}(\epsilon_1) w_{0\nu}(\epsilon_1)\right]$$

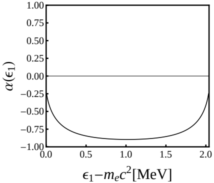
where $\mathcal{N}_{0\nu} = g_A^4 \left(\frac{\langle m_\nu \rangle}{m_e}\right)^2 |M_{GT}^{(0\nu)} - \left(\frac{g_V}{g_A}\right)^2 M_F^{(0\nu)} + M_T^{(0\nu)}|^2$

- and the angular correlation: $\alpha(\epsilon_1) = \frac{f_{11}^{(1)}(\epsilon_1)}{f_{11}^{(0)}(\epsilon_1)} = \frac{dG_{0\nu}^{(1)}/d\epsilon_1}{dG_{0\nu}/d\epsilon_1}$



- Current $0\nu\beta^-\beta^-$ PSFs (red) compared to previous calculations (blue)
- Influence of using the exact electron wave functions is seen especially for heavier isotopes





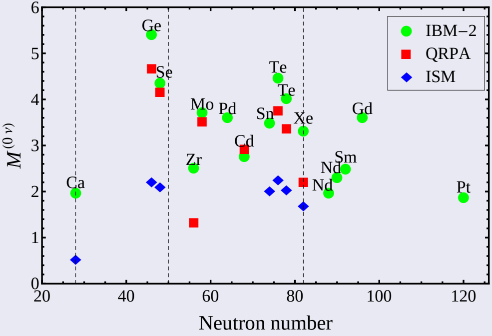
- Example of single electron spectrum: ^{76}Ge $\rightarrow$ ^{76}Se decay
- Example of angular correlations: ^{76}Ge $\rightarrow$ ^{76}Se decay

$0\nu\beta^-\beta^-$: Nuclear matrix elements

- Most used models:
  - QRPA: Results depend on fine-tuning of the interaction, especially near the spherical- deformed transition, for example ^{150}Nd.
  - ISM: Cannot address nuclei with many particles in the valence shells, for example ^{150}Nd, due to the exploding size of the Hamiltonian matrices ($> \mathbf{10^9}$).
- Recent advances
  - Development of a program to compute $\mathbf{0\nu\beta\beta}$ and $\mathbf{2\nu\beta\beta}$ nuclear matrix elements in the closure approximation within the framework of the microscopic Interacting Boson Model (IBM-2)
    - Can be used in any nucleus and thus all nuclei of interest can be calculated within the same model

$0\nu\beta^-\beta^-$: Nuclear matrix elements

$$M^{(0\nu)} = M_{GT}^{(0\nu)} - \left(\frac{g_V}{g_A}\right)^2 M_F^{(0\nu)} + M_T^{(0\nu)}$$



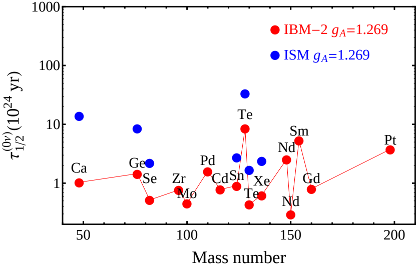
- Comparison of QRPA*, ISM** and IBM-2 matrix elements for light neutrinos with Jastrow SRC
- The ISM is a factor of (approximately) two smaller than both the IBM-2 and QRPA
  - This could suggest that $g_{A,eff}$ is the same for both modes
- The agreement between IBM-2 and QRPA is not only for the overall matrix element but also for the individual pieces, F, GT, T

* F. Šimkovic *et al.*, Phys. Rev. C **77**, 045503 (2008).
** E. Caurier *et al.*, Phys. Rev. Lett. 100, 052503 (2008).

$0\nu\beta^-\beta^-$:Results with g_A and $g_{A,eff}$

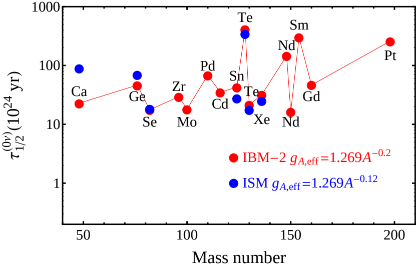
- Predictions for $0\nu\beta^-\beta^-$ half-lives with $\langle m_\nu \rangle = 1eV$

$g_A = 1.269$



- bad agreement between IBM-2 and ISM

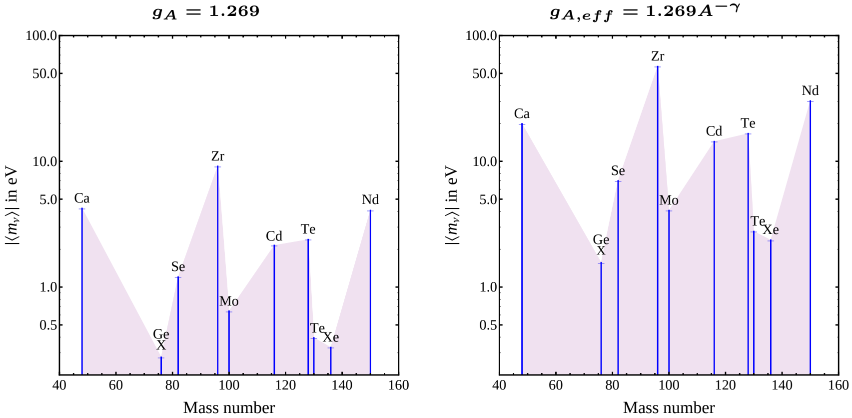
$g_{A,eff} = 1.269A^{-\gamma}$



- good agreement between IBM-2 and ISM
- Half-lives over ten times longer!

$0\nu\beta^-\beta^-$: Results with g_A and $g_{A,eff}$

- Limits on average light neutrino mass from current experimental $0\nu\beta^-\beta^-$ half-life limits*



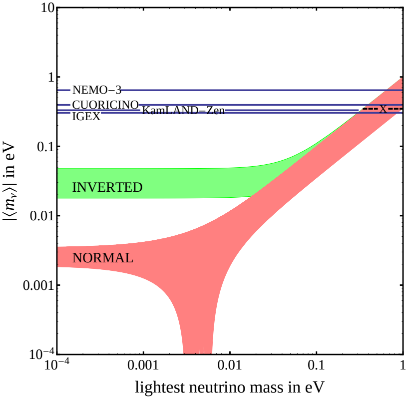
- Best candidates at the moment ^{76}Ge, ^{100}Mo, ^{130}Te and ^{136}Xe

* A.S. Barabash, Phys. Atom. Nucl. **74**, 603 (2011).

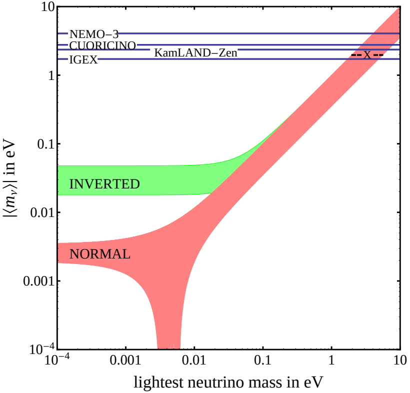
$0\nu\beta^-\beta^-$: Results with g_A and $g_{A,eff}$

- Current limits to $\langle m_\nu \rangle$ from CUORICINO, IGEX, NEMO-3 and KamLAND-Zen

$g_A = 1.269$



$g_{A,eff} = 1.269A^{-\gamma}$



IGEX: C. E. Aalseth *et al.*, Phys. Rev. D **65**, 092007 (2002).
NEMO-3: R. Arnold, *et al.*, Nucl. Phys. A **765**, 483 (2006).
CUORICINO: C. Arnaboldi *et al.*, Phys. Rev. C **78**, 035502 (2008).
KamLAND-Zen: A. Gando *et al.*, arXiv:1201.4664vi [hep-ex] (2012).
X: H.V. Klapdor-Kleingrothaus *et al.*, Phys. Lett. B **586**, 198 (2004).

Conclusions

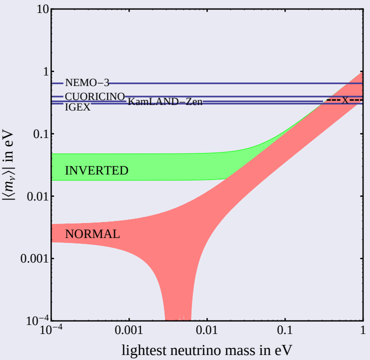
- Complete and improved calculation of phase space factors for $2\nu\beta^-\beta^-$ and $0\nu\beta^-\beta^-$ decay
  - including half-lives, single electron spectra, summed electron spectra and electron angular correlations
- Improvement: exact Dirac wave function with finite nuclear size and electron screening
- Improvement: first excited 0^+ state, not just ground state
- Error analysis included to maximize the feasibility
- The analysis of the $g_{A,eff}$ for the case of IBM-2 NMEs
- Predictions for half-lives and limits for average light neutrino masses using IBM-2 matrix elements

Outlook

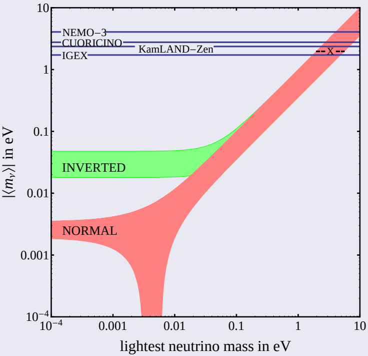
- Complete and improved calculations for $2\nu\beta^+\beta^+$ and $0\nu\beta^+\beta^+$ decay phase space factors, as well as for $2\nu EC\beta^+$, $2\nu ECEC$ and $0\nu EC\beta^+$ are ready, $0\nu ECEC$ needs more work
- Similar analysis to these modes that was done for $\beta^-\beta^-$

THANK YOU!

$g_A = 1.269$



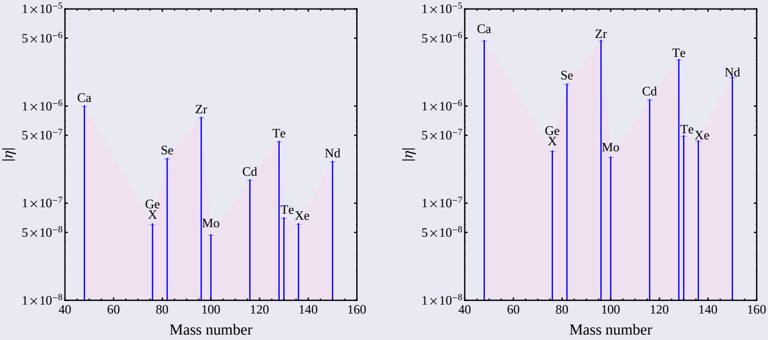
$g_{A,eff} = 1.269 A^{-\gamma}$



Estimate of uncertainties introduced to PSF

$\mathbf{2\nu}$	Q-value	$\mathbf{10 \times \delta Q/Q}$
	Radius	**0.5%**
	Screening	**0.10%**
	$\langle E_N \rangle$	model dependent
$\mathbf{0\nu}$	Q-value	$\mathbf{3 \times \delta Q/Q}$
	Radius	**7%**
	Screening	**0.10%**
	$\langle E_N \rangle$	-

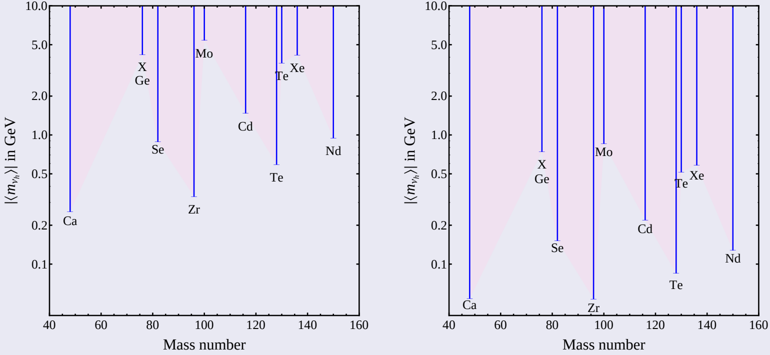
Limits on $|\eta|$ with g_A and $g_{A,eff}$



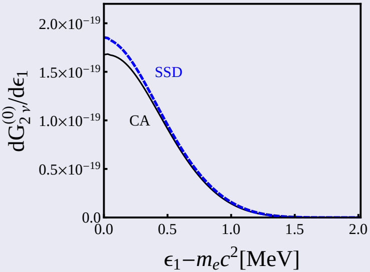
Limits on $\langle m_{\nu_h} \rangle$ with g_A and $g_{A,eff}$

$$\eta = \frac{M_W^4}{M_{WR}^4} \sum_{k=heavy} \left(V_{ek_h}\right)^2 \frac{m_p}{m_{k_h}},$$

where M_W is the mass of the W-boson, $M_W = 80.41 \pm 0.10$GeV, M_{WR} is the mass of WR-boson, assumed* to be $M_{WR} = 3.5$TeV and $V = (M_{WR}/M_W)^2\, U$. The ratio $(M_W/M_{WR})^4$ is then 2.75×10^{-7}



* V. Tello *et al.*, Phys. Rev. Lett. **106**, 151801 (2011).



2.0×10^{-19}
1.5×10^{-19}
1.0×10^{-19}
0.5×10^{-19}
0.0
SSD
CA
0.0
0.5
1.0
1.5
2.0
$dG^{(0)}_{2\nu}/d\epsilon_1$
$\epsilon_1 - m_e c^2$ [MeV]

The functions $f_{11}^{(0)}$ and $f_{11}^{(1)}$ are defined as

$$\begin{aligned} f_{11}^{(0)} &= |f^{-1-1}|^2 + |f_{11}|^2 + |f^{-1}{}_1|^2 + |f_1{}^{-1}|^2, \\ f_{11}^{(1)} &= -2\mathrm{Re}[f^{-1-1} f_{11}^* + f^{-1}{}_1 f_1{}^{-1*}]. \end{aligned} \qquad (1)$$

with

$$\begin{aligned} f^{-1-1} &= g_{-1}(\epsilon_1) g_{-1}(\epsilon_2), \\ f_{11} &= f_1(\epsilon_1) f_1(\epsilon_2), \\ f^{-1}{}_1 &= g_{-1}(\epsilon_1) f_1(\epsilon_2), \\ f_1{}^{-1} &= f_1(\epsilon_1) g_{-1}(\epsilon_2). \end{aligned} \qquad (2)$$

$$\begin{aligned} g_{-1}(\epsilon) &= \int_0^\infty w(r) g_{-1}(\epsilon, r) r^2 dr, \\ f_1(\epsilon) &= \int_0^\infty w(r) f_1(\epsilon, r) r^2 dr. \end{aligned} \qquad (3)$$

In approximation (I) we use the weighing function $w(r) = \delta(r - R)/r^2$ in which case

$$\begin{aligned} g_{-1}(\epsilon) &= g_{-1}(\epsilon, R) \\ f_1(\epsilon) &= f_1(\epsilon, R) \end{aligned} , \qquad \text{(I)} \tag{4}$$

that is the electron wave functions are evaluated at the nuclear radius $r = R$. This is the simplest approximation and is commonly used in single-β decay. In approximation (II) we use the weighing function $w(r) = 3/R^3$ for $r \leq R$ and $w(r) = 0$ for $r > R$ (an uniform distribution of radius R). This is not a good approximation, since the inner states cannot decay due to Pauli blocking and the decay occurs at the surface of the nucleus.

$$\begin{aligned} g_{-1}(\epsilon) &= \frac{3}{R^3}\int_0^R g_{-1}(\epsilon, r) r^2 dr \\ f_1(\epsilon) &= \frac{3}{R^3}\int_0^R f_1(\epsilon, r) r^2 dr \end{aligned} . \qquad \text{(II)} \tag{5}$$

The third and most accurate approximation (III) is that in which the weighing function is the square of the wave function, $R_{nl}(r)$, of the nucleon undergoing the decay,

$$\begin{aligned} g_{-1}(\epsilon) &= \int_0^\infty |R_{nl}(r)|^2 \, g_{-1}(\epsilon, r) r^2 dr \\ f_1(\epsilon) &= \int_0^\infty |R_{nl}(r)|^2 \, f_1(\epsilon, r) r^2 dr \end{aligned} . \qquad \text{(III)} \tag{6}$$

The approximation (III) essentially amounts to an evaluation of $g_{-1}(\epsilon)$ and $f_1(\epsilon)$ at a radius $\sqrt{\langle r^2 \rangle_{nl}}$. For harmonic oscillator wave functions one has

$$\left\langle r^2 \right\rangle_{nl} = b^2 \left(2n + l + \frac{3}{2} \right) . \tag{7}$$

This approximation has the disadvantage that it must be done separately for each nucleus.