Algebra preserving metric deformations of a conformal group manifold and degeneracies in hadron spectra

M. Kirchbach

IF-UASLP

Dedicated to the 70. birthday of Francesco Yachello

Contents

Introduction

Group theoretical methods

Powerful tool in classifying the excitation modes of composite systems in terms of irreducible representations

Quantum Rigid Rotator:

$$\frac{\hbar^2}{2\mathcal{I}}\mathbf{L}^2 Y_l^m(\theta,\varphi) = \frac{\hbar^2}{2\mathcal{I}} l(l+1) Y_l^m(\theta,\varphi), \quad \mathcal{I} = \mu a^2$$

Spectrum characterized by a $(2l+1)$-multiplicity of the states in a level $\mathbf{L}^2$
Casimir invariant of the $so(3)$ algebra in terms of $\mathbf{r}^2$ conserving operators

$$\mathbf{L}^2 = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

Effectively, the Rigid Rotator describes free geodesic motion of a scalar particle on the SO(3) group manifold S^2

Introduction

The curious case of the cotangent hindered 3D rigid rotator (dimensionless units):

$$\left[\mathbf{L}^2 - 2b\cot\theta\right]\mathcal{X}_t^{|\widetilde{m}|}(\theta,\varphi) = \left[t(t+1) - \frac{b^2}{t(t+1)+\frac{1}{4}}\right]\mathcal{X}_t^{|\widetilde{m}|}(\theta,\varphi)$$

whose wave functions are

$$\begin{aligned}\mathcal{X}_t^{|\widetilde{m}|}(\theta,\varphi) &= F_t^{|\widetilde{m}|}(\theta)e^{i\widetilde{m}\varphi}\\ t=0,1,2,.., &\quad |\widetilde{m}|=0,1,2,..,t\end{aligned}$$

The spectrum of the hindered rotator has same $(2t+1)$ fold level multiplicity as the free rotator, though

$$\left[\mathbf{L}^2, \cot\theta\right] \neq 0$$

Introduction

$$F_t^{|\widetilde{m}|}(\theta) \quad = \quad N_{t|\widetilde{m}|} \sin^t \theta e^{-\frac{b\theta}{t+\frac{1}{2}}} R_{n=t-|\widetilde{m}|}^{\frac{2b}{t+\frac{1}{2}}, -(t-\frac{1}{2})}(\cot\theta), \quad t = n + |\widetilde{m}|$$

$R_n^{\alpha,\beta}(\cot\theta)$ are Romanovski polynomials,

$$(1+x^2)\frac{\mathrm{d}^2 R_n^{\alpha,\beta}}{\mathrm{d}x^2} + 2\left(\frac{\alpha}{2} + \beta x\right)\frac{\mathrm{d}R_n^{\alpha,\beta}}{\mathrm{d}x} - n(2\beta + n - 1)R_n^{\alpha,\beta} = 0$$

obtained from the weight function

$$\omega^{\alpha,\beta}(x) = (1+x^2)^{\beta-1}\exp(-\alpha\cot^{-1}x)$$

by means of the Rodrigues formula:

$$R_n^{\alpha,\beta}(x) = \frac{1}{\omega^{\alpha,\beta}(x)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1+x^2)^n\omega^{\alpha,\beta}(x)\right]$$

The parameters α and β are

$$\alpha = \frac{2b}{t+\frac{1}{2}}, \qquad \beta = -\left(t + \frac{1}{2}\right) + 1, \quad t = n + |\widetilde{m}|$$

Introduction

- The $\cot\theta$ perturbed RR on S^2 describes rotational(like) bands with an anomalous large gap between the ground state and the first excited level
- In anticipation of the next section: The trigonometric Rosen-Morse potential on S^2 has an so(3) symmetry algebra represented by operators unitarily inequivalent to those conserving $\mathbf{r}^2$ by rotations

Alvarez-Castillo, Compean, M.K. Mol.Phys. **109**, *1477 (2011)*

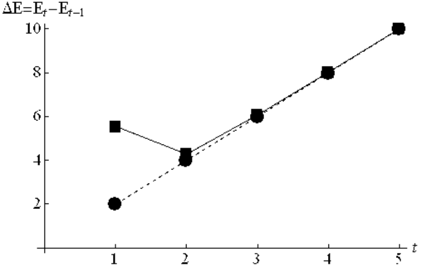


Figure: Anomalous $\Delta E = (E_1 - E_0)$ gap in the rotational(like) band of the cotangent hindered 3D rigid rotator

The potential algebra concept

The Pöschl–Teller potential

$$V_{PT} = \frac{A(A+\alpha)}{\cosh^2 r}$$

serves as a point of departure for the construction of various hyperbolic potentials. In all the cases the corresponding Schrödinger Hamiltonians can be cast as Casimir invariants of an $su(1,1) \sim so(1,2)$ algebra. Then the labels of the $su(1,1) \sim so(1,2)$ irreps determine energy spectrum and wave functions.

Alhassid, Gürsey, Yachello, Phys.Rev.Lett. **50** *(1983)*

Closely related Eckart (ET) potential, $V_{ET} = \frac{A(A-\alpha)}{\sinh^2 r}$:
Free geodesic motion (in dimensionless units) of a scalar particle on the upper sheet, $\mathbf{H}^2_+$, of a two-sheeted hyperboloid

$$\begin{aligned}
H_{ET} &= -C_{so(1,2)} = -\frac{1}{\sinh\eta}\frac{\partial}{\partial\eta}\sinh\eta - \frac{\partial^2_\varphi}{\sinh^2\eta} \\
H_{ET}\mathcal{Y}_l^m(\eta,\varphi) &= -l(l+1)\mathcal{Y}_l^m(\eta,\varphi) \\
\mathcal{Y}_l^m(\eta,\varphi) &= P_l^m(\cosh\eta)e^{im\varphi}, \quad \eta\in(-\infty,+\infty) \\
C_{so(1,2)}|lm> &= l(l+1)|lm>, \quad J_z|lm>= m|lm>
\end{aligned}$$

Symmetry breaking by metric deformation of a group manifold

Motion on $\mathbf{H}^2_+$ perturbed by $\coth\eta$

$$\begin{aligned} -\left(C_{su(1,1)} + 2b\coth\eta\right)\mathcal{X}_t^m(\eta,\varphi) &= \left(-t(t+1) - \frac{b^2}{\left(t+\frac{1}{2}\right)^2}\right)\mathcal{X}_t^m(\eta,\varphi) \\ \mathcal{X}_t^m(\eta,\varphi) &= \mathcal{F}_t^m(\eta)e^{im\varphi} \end{aligned}$$

is characterized by same $(2t+1)$ multiplicity of the states in a level (same degeneracies) as the unperturbed motion despite of non-commutativity $C_{so(1,2)}$ with the perturbance (hyperbolic version of the $\cot\theta$ perturbed RR on S^2)

Symmetry breaking by metric deformation of a group manifold

Explanation:

- *Wu, Alhassid, J.Math.Phys.* **31**, *557 (1990)*

$$\begin{aligned}\mathbf{F}(\eta(r))\mathcal{C}_{so(1,2)}\mathbf{F}^{-1}(\eta(r)) &= \widetilde{\mathcal{C}}_{so(1,2)}(r)\\ \mathbf{F}(\eta(r)) &= \sqrt{\frac{\sinh 2g(r)}{g'(r)}}, \quad \tanh^2 g(r) = z = e^{-r}\end{aligned}$$

with a non-unitary **F**, transforms H_{ET} into a Natanzon Ham. in the z variable, and to the central Eckart potential problem in ordinary 3D flat position space in the r variable, thus revealing $so(1,2)$ as the **symmetry algebra** of the Eckart potential

- *Leija-Martinez, Alvarez-Castillo, M.K., Symm.Integr.Geom.:Meth.Apl.(SIGMA)* **7** *(2011)*

$$\begin{aligned}
\left(C_{so(1,2)} + 2b\coth\eta\right)\mathcal{X}_t^{\widetilde{m}}(\eta,\varphi) &= \mathbf{F}(\eta)C_{so(1,2)}\mathbf{F}^{-1}(\eta)\mathcal{X}_t^{\widetilde{m}}(\eta,\varphi)\\
&= \widetilde{C}_{so(1,2)}\mathcal{X}_t^{\widetilde{m}}(\eta,\varphi)\\
\mathbf{F}(\eta,\varphi) &= e^{-\frac{\alpha_t\eta}{2}}\mathbf{A}^{(t)}(\varphi), \quad \alpha_t = \frac{2b}{t+\frac{1}{2}}\\
\mathcal{X}_t^{\widetilde{m}}(\theta,\varphi) &= e^{-\frac{\alpha_t\eta}{2}}\sum_{m=\widetilde{m}}^{m=t}\mathbf{A}^{(t)}_{(\widetilde{m}+1)(m+1)}(\varphi)\mathcal{Y}_t^m(\eta,\varphi)
\end{aligned}$$

$\widetilde{C}_{so(1,2)}$ describes a free geodesic motion on an exponentially rescaled hyperboloid whose isometry algebra continues being $so(1,2)$ though represented by operators unitarily inequivalent to those conserving $\mathbf{s}^2 = z^2 - x^2 - y^2$

Symmetry breaking by metric deformation of a group manifold

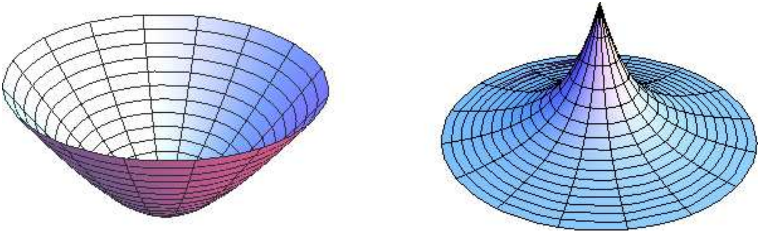


Figure: Breaking of the pseudo-rotational symmetry at the level of the metric. A regular $\mathbf{H}^2_+$ hyperboloid associated with the so(1,2) scalar, $(z^2 - x^2 - y^2)$ (left), and its non-unitary deformation, $\exp(-\alpha_{(t=0)}\eta/2)(z^2 - x^2 - y^2)$ (right) with $\alpha_{(t=0)} = 4b$, and $\eta = \coth^{-1}\frac{z}{\sqrt{x^2+y^2}}$. Schematic presentation for $b = 1$.

Observations:

- Potential algebras produce isometry copies to group manifolds by metric deformations
- Free geodesic motion on the copy is equivalent to a motion on the group manifold perturbed by a potential
- As long as the wave functions of the free and perturbed motions are eigenfunctions of a Casimir invariant of same algebra, no matter whether or not represented by operators that conserve $\mathbf{r}^2/\ \mathbf{s}^2$ etc., free and perturbed motions are described by means of wave functions transforming under same irreps of the algebra under consideration. They carry same quantum numbers and degeneracies though different level splittings.
- The symmetry breaks at the level of the metric of the original group manifold, but is preserved at the level of the isometry algebra of the deformed manifold
- The above symmetry breakings can be viewed as brought about by mass scales, associated with the potential strengths

Candidate for a QCD potential of so(2,4) symmetry algebra

AdS_5/CFT_4 gauge gravity correspondence:

A weakly coupled string theory at the boundary (null-ray cone) of the one-sheeted 6D hyperboloid AdS_5 appears dual to a conformal field theory (CFT_4) in (1+3) dimensions, associated with QCD at high temperatures (HT).

- *Maldacena, Phys. Rev. Lett.* **80**, *4859 (1998)*

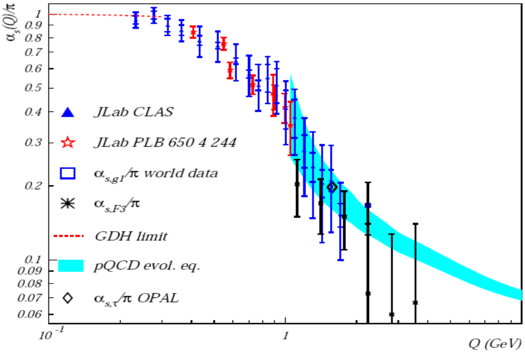
  The conformal SO(2,4) group manifold appropriate for HT QCD is the compactified Minkowski spacetime $\mathcal{M}^{1+3} = S^1 \otimes S^3$ *with* $T \sim 1/R^2$ *with* R *the* S^3 *radius, and* $T >> \Lambda_{QCD}$

- *Witten, Adv. Theor. Math. Phys.* **2**, *233 (1998)*

Candidate for a QCD potential of so(2,4) symmetry algebra

QCD is conformal in two regimes:

- In the UV regime of asymptotic freedom, $\alpha \rightarrow 0$, described by means of Light Cone techniques (Brodsky, Terramónd)
- In the IR regime beneath the conformal window where the strong coupling walks towards a fixed value, $\alpha \rightarrow$const.



Intact conformal group symmetry and degeneracies

The line interval on the unit S^3 hypersphere reads,

$$ds^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)$$

The free geodesic motion on S^3 is described by means of the conformal Laplacian, $\mathbf{L}_{S^3}$

$$\mathbf{L}_{S^3} = -\frac{1}{R^2}\Delta_{S^3} + \mathcal{R}_{S^3}, \quad \mathcal{R}_{S^3} = \frac{1}{R^2}, \quad R^2 = 1$$

$$-\frac{1}{R^2}\Delta_{S^3} = -\frac{1}{R^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} + 2\cot\chi\frac{\mathrm{d}}{\mathrm{d}\chi} - \frac{\mathbf{L}^2}{\sin^2\chi}\right) = \frac{1}{R^2}\mathcal{K}^2$$

$\mathcal{K}^2$ is the operator of the squared four-dimensional (4D) angular momentum which acts as a Casimir invariant of the isometry algebra $so(4)$ of S^3, and $\mathcal{R}_{S^3}$ is the conformal curvature

Intact conformal group symmetry and degeneracies

Specifically, on the unit hypersphere, the spectral problem of $\mathbf{L}_{S^3}$ is

$$\begin{aligned} \mathbf{L}_{S^3} Y_{Klm}(\vec{\Omega}) &= \left(\mathcal{K}^2 + 1\right) Y_{Klm}(\vec{\Omega}) = (K+1)^2 Y_{Klm}(\vec{\Omega}), \\ \vec{\Omega} &= (\chi, \theta, \varphi) \\ Y_{Klm}(\vec{\Omega}) = \mathcal{S}_K^l(\chi) Y_l^m(\theta,\varphi), & \quad \mathcal{S}_K^l(\chi) = \sin^l \chi \mathcal{G}_{K-l}^{l+1}(\cos\chi), \\ K \in [0,\infty), \quad l \in [0,K], & \quad m \in [-l,+l]. \end{aligned}$$

This is the spectrum corresponding to an intact conformal group symmetry and describes 4D rotational bands of $(K+1)^2$-fold degeneracies of the states in a level

Intact conformal group symmetry and degeneracies

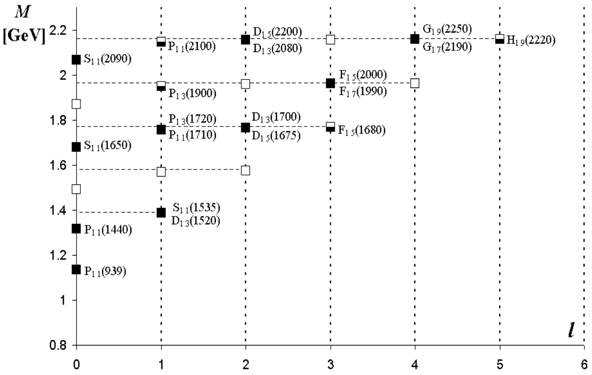


Figure: The N spectrum.Empty squares mark “missing” states. The levels are Rarita-Schwinger fields, $\Psi_{\mu_1 \dots \mu_K}$, with K=0,1,2,..5 (all lower spins appear parity doubled).

M.K., Compean, Phys. Rev. D **82** *(2010)*

Intact conformal group symmetry and degeneracies

Observations:

- Pronounced conformal 4D rot. bands of $(K+1)^2$-fold degeneracies
- Anomalously large gap between the ground state (K=0), and the first excited group of three (K=1)
- Degeneracies signal $so(2,4) \supset so(4)$ symmetry algebra of the QCD potential, while the anomalous gap signals global group symmetry breaking by a scale, taken as the dilaton mass.

Motion on S^3 hindered by the "curved Coulomb potential"

Introducing a potential, breaks the conformal curvature

$$\hbar^2 c^2 \mathcal{R}_{S^3} \longrightarrow \hbar^2 c^2 \widetilde{\mathcal{R}}_{S^3} = \frac{\hbar^2 c^2}{R^2} + V(\vec{\Omega})$$

by a mass scale, associated with the potential strength. For $V(\chi) = -2B\cot\chi$ with $B = b\hbar^2 c^2/R^2$, and a dimensionless b

$$\frac{\hbar^2 c^2}{R^2}\left[-\Delta_{S^3} + 1 - 2b\cot\chi\right]\Psi_{K\widetilde{l}\widetilde{m}}(\vec{\Omega}) = \frac{\hbar^2 c^2}{R^2}\left[(K+1)^2 - \frac{b^2}{(K+1)^2}\right]\Psi_{K\widetilde{l}\widetilde{m}}(\vec{\Omega})$$

$$\Psi_{K\widetilde{l}\widetilde{m}}(\vec{\Omega}) = e^{\frac{\alpha_K \chi}{2}}\psi_K^{\widetilde{l}}(\chi) Y_{\widetilde{l}}^{\widetilde{m}}(\theta,\varphi)$$

$$\psi_K^{\widetilde{l}}(\chi) = \sin^K \chi R_{K-\widetilde{l}}^{\alpha_K,\beta_K-1}(\cot\chi)$$

$$\alpha_K = -\frac{2b}{K+1}, \qquad \beta_K = -K$$

Motion on S^3 hindered by the "curved Coulomb potential"

We find 4D rotational bands with an anomalous gap between the K=0 and K=1 states. The manifest $so(4)$ symmetry algebra of the $\cot\chi$ potential ("curved" Coulomb) has been constructed from the explicit decompositions of $\psi_K^{\widetilde{l}}(\chi)$ in the basis of $\sin^l\chi\,\mathcal{G}_{K-l}^{l+1}(\cos\chi)$, the quasi-radial parts of the hyper-spherical harmonics , $Y_{Klm}(\chi,\theta,\varphi)$

Pallares-Rivera, M.K., J.Phys. A:Math.Theor. **44**,*445302 (2011)*

$$\begin{aligned}
\left(C_{so(4)} + 2b\cot\chi\right)\Psi_{K\widetilde{l}\widetilde{m}}(\vec{\Omega}) &= \mathbf{F}(\vec{\Omega})C_{so(4)}\mathbf{F}^{-1}(\vec{\Omega})\Psi_{K\widetilde{l}\widetilde{m}}(\vec{\Omega})\\
\mathbf{F}(\vec{\Omega}) &= e^{-\frac{\alpha_K\chi}{2}}\mathbf{A}^{(K)}(\theta,\varphi)\\
\Psi_{K\widetilde{l}\widetilde{m}}(\vec{\Omega}) &= e^{-\frac{\alpha_K\chi}{2}}\sum_{l=\widetilde{l}}^{l=K}\mathbf{A}^{(K)}_{(\widetilde{l}+1)(l+1)}(\theta,\varphi)Y_{Kl\widetilde{m}}(\vec{\Omega})\\
\alpha_K &= \frac{2b}{K+1}
\end{aligned}$$

Motion on S^3 hindered by the "curved Coulomb potential"

The "curved" Coulomb (trigonometric Rosen-Morse in flat space) is a confinement potential of finite range

$$-2b\cot\frac{\widehat{r}}{R} = -\frac{2bR}{\widehat{r}} + \frac{4b}{3R}\,\widehat{r} + ..., \qquad \text{for} \quad \chi = \frac{\widehat{r}}{R}$$

and can be viewed as an the exactly solvable extension of the Cornell potential predicted by QCD where $\widehat{r}$ denotes the geodesic distance on S^3, and R was the S^3 radius

Compean, M.K. Eur.Phys.J. A **33**, *1 (2007)*

More recently, the above confinement potential has been employed to study degeneracies in the spectrum of the respective 1D Hamiltonian which occur upon deformations of its super-symmetric algebra

Aleixo, Balantekin, J.Phys.A:Math.Theor. **44**, *365303 (2011)*

Motion on S^3 hindered by the "curved Coulomb potential"

The $\cot\chi$ perturbed motion on S^3 is equivalent to free motion on an so(4) isometry copy of S^3 of the deformed metric

$$\mathrm{d}^2\widetilde{s} \quad = \quad e^{-b\chi}\left((1+b^2/4)\mathrm{d}^2\chi + \sin^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}^2\varphi\right)\right).$$

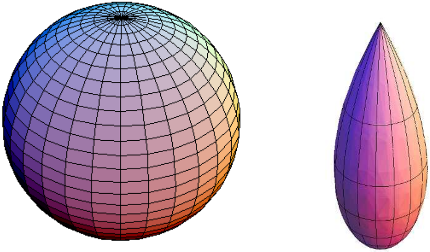


Figure: Deformation of the spherical metric, $|Y_{000}(\chi, 0\,\varphi)|$, (left) versus the exponentially scaled one, $|\widetilde{Y}_{000}(\chi, 0, \varphi)|$ (right) for b=1.

Degeneracies of excited light flavor mesons

We fit the b and R parameters to the Crystal Barrel data on the spectra of the high lying unflavored mesons

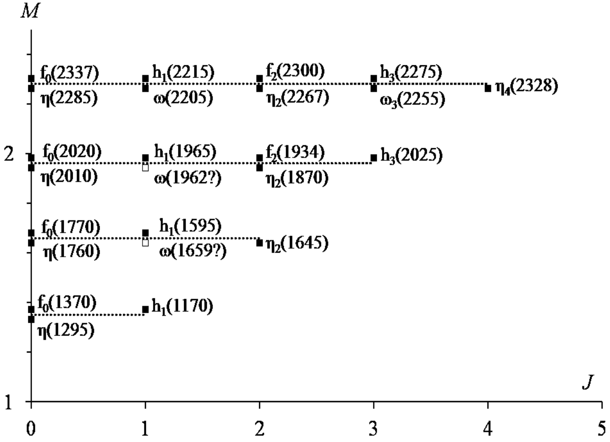


Figure: Excitations of the $\eta(1295)$ meson

Degeneracies of excited light flavor meson

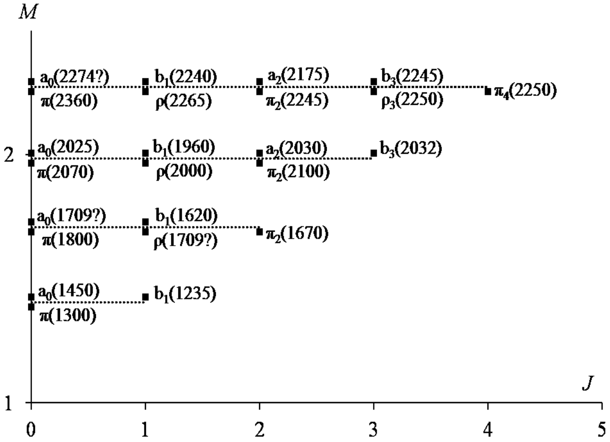


Figure: Excitations of the $\pi(1300)$ meson

Degeneracies of excited mesons

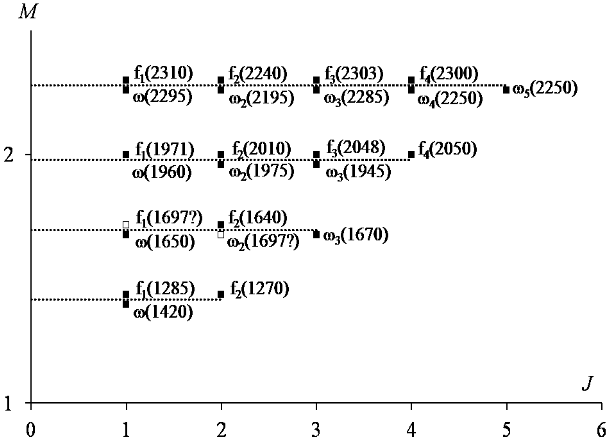


Figure: Excitations of the $\omega(1420)$ meson

Degeneracies of excited light flavor mesons

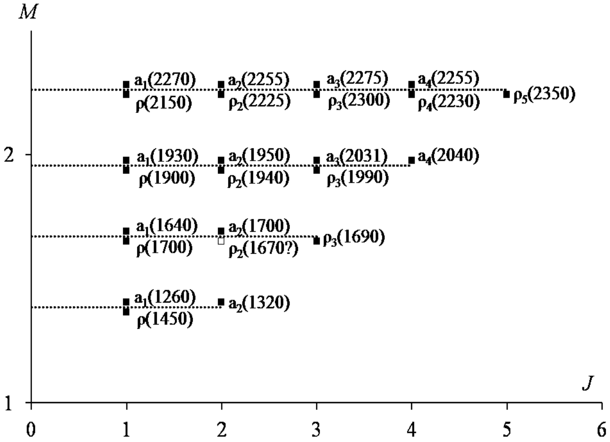


Figure: Excitations of the $\rho(1450)$ meson

Degeneracies of excited light flavor mesons

Results:

- $b = 3.27, \quad m_d = \hbar c\sqrt{b}/R$=673.7 MeV
- $R = 0.5290$ fm $\quad$ T=$\hbar c/R = 373$ MeV $> \Lambda_{QCD}$=173 MeV

Summary

- We suggest a scenario in which the conformal symmetry of QCD following from the gauge-gravity duality is broken by the dilaton mass at the level of the metric of the group manifold, and conserved locally as symmetry algebra of the potential
- As a candidate for a QCD potential in the IR we suggest the "curved" Coulomb potential which has an so(4) symmetry algebra and explains the degeneracies in the spectra of the unflavored hadrons, both baryons and mesons in the mass range $M^* \in (1400, 2500)$ MeV
- We conclude that conformal symmetry is likely to be a partial dynamical symmetry of QCD at least in the mass range of $M^* \in (1400, 2500)$ MeV and in the spirit of

  Garcia-Ramos, Leviatan, Van Isacker,Phys. Rev. Lett. **102**, *(2009)*

Summary

- We conjecture equality between the potential strength and the dilaton mass and obtain the quite realistic value of $m_d \sim 700$ MeV
- The potential under investigation furthermore allows for an exact Fourier transform to momentum space where it takes the form of

$$\Pi(|\mathbf{q}|) \quad = \quad -b\frac{\frac{\sin^2|\mathbf{q}|}{2}}{\frac{\mathbf{q}^2}{2}}$$

Compean, M.K., J.Phys.A:Math.Theor. **44**, *015304 (2011)*

- Our scheme suggests a description of hadron spectra by means of a generalization of the algebraic Hamiltonians of the type

$$\mathcal{H} \quad = \quad a_{-1}C^{-1}_{so(4)} + a_1 C_{so(4)} + a_2 C^2_{so(4)} + ... + bC_{so(3)}$$

Iachello, Mukhopadhyay, Zhang, Phys. Rev. D **44** *(1991)*
according to $C_{so(4)} \rightarrow \mathbf{F} C_{so(4)} \mathbf{F}^{-1}$ for appropriate non-unitary $\mathbf{F}$.