Happy Birthday Franco!

The origin of order in random matrices... with symmetries

Calvin W. Johnson
Dept of Physics, San Diego State University

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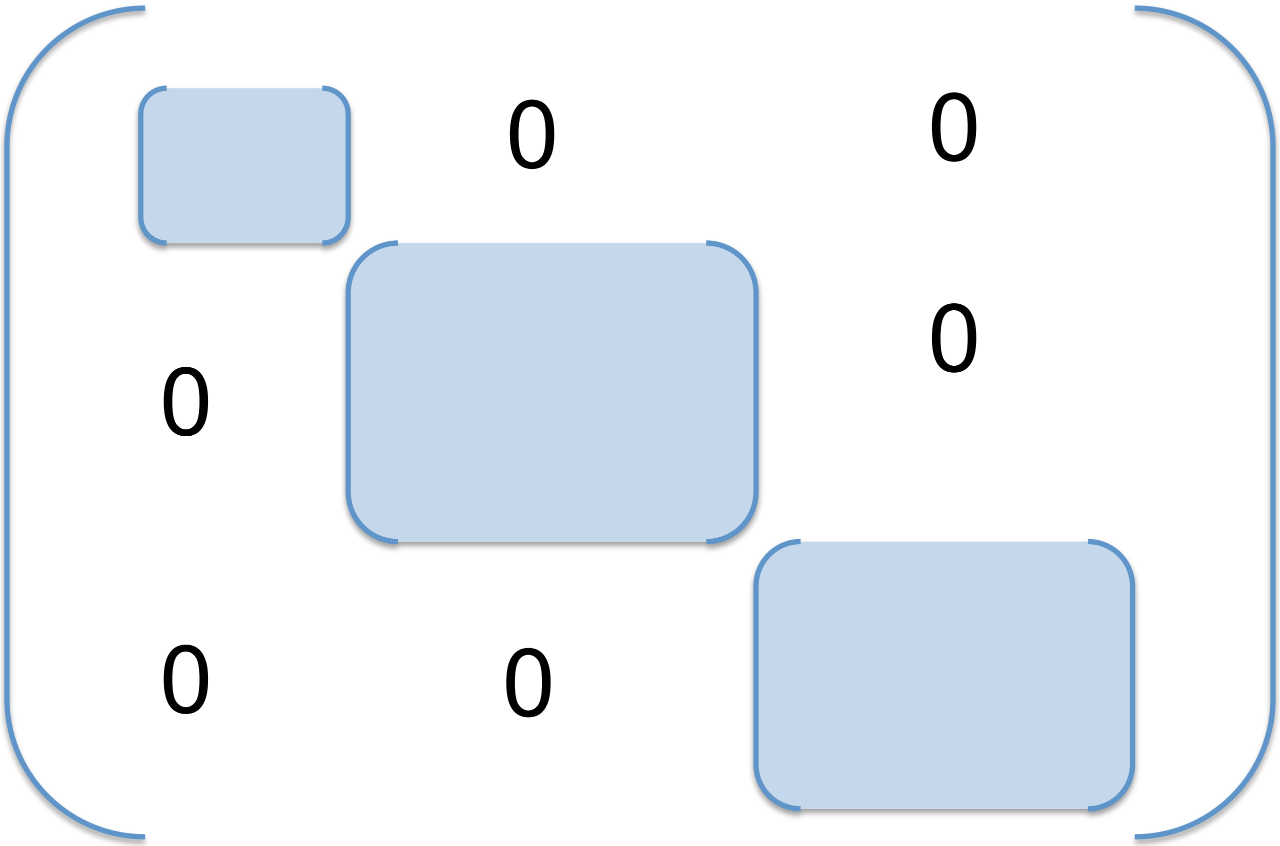
"Beauty in Physics" Cocoyoc May 2012

"The most beautiful result
in mathematical physics..."

Emmy Noether's theorem:

*A symmetry leads to
a conserved quantity*

If the Hamiltonian commutes with the generator(s) of a symmetry, then we can write the Hamiltonian as *block diagonal* with the blocks (subspaces) defined by the irreps of the symmetry group:



But there is a mystery that we seldom think about: the ground state is almost *always* dominated by the "most symmetric" irrep (often one of lowest dimension, too)

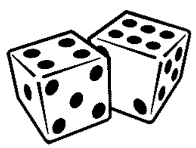
E.g., *translational invariance* leads to *conserved momentum*....
in QM state exp(ipx).... lowest *energy* state has $p=0$ (also most symmetric)

rotational invariance leads to *conserved angular momentum*....
lowest *energy* state is usually L=0 (or J= 0) even in many-body systems (also irrep with lowest dimension (2J+1) irrep)

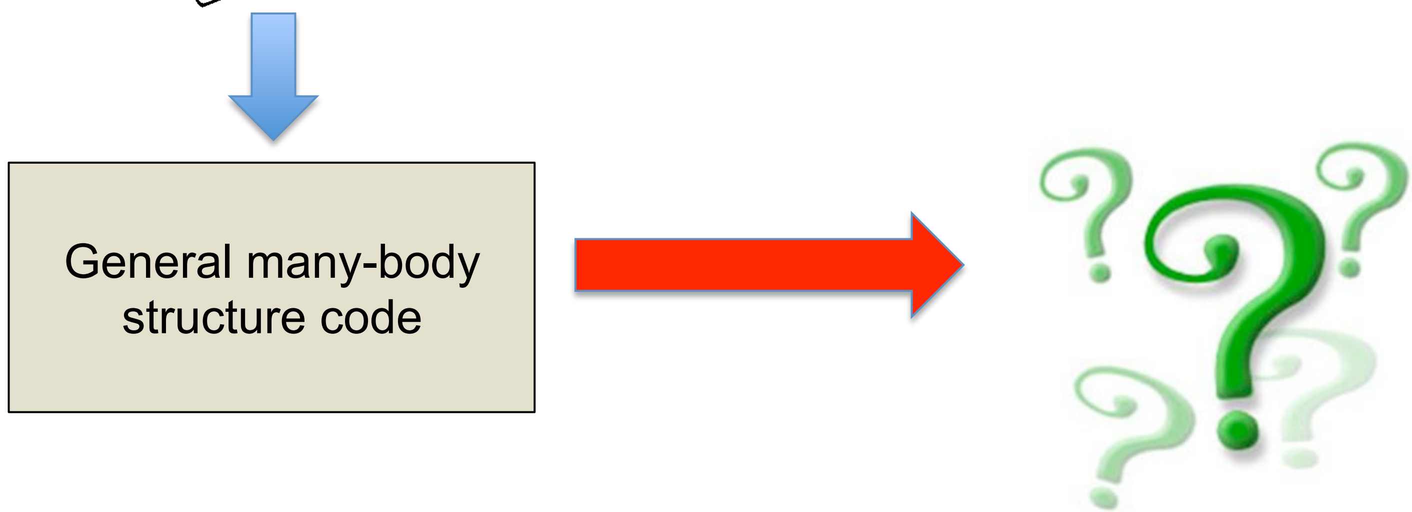
Of course we can "explain" the simple cases because the Hamiltonian is quadratic in momentum, p^2 ....

...only this persists even when we erase any such argument, e.g. with *random interactions*

A numerical experiment:



Draw interaction from "two-body random ensemble"



48Ca in *pf* shell (8 neutrons)

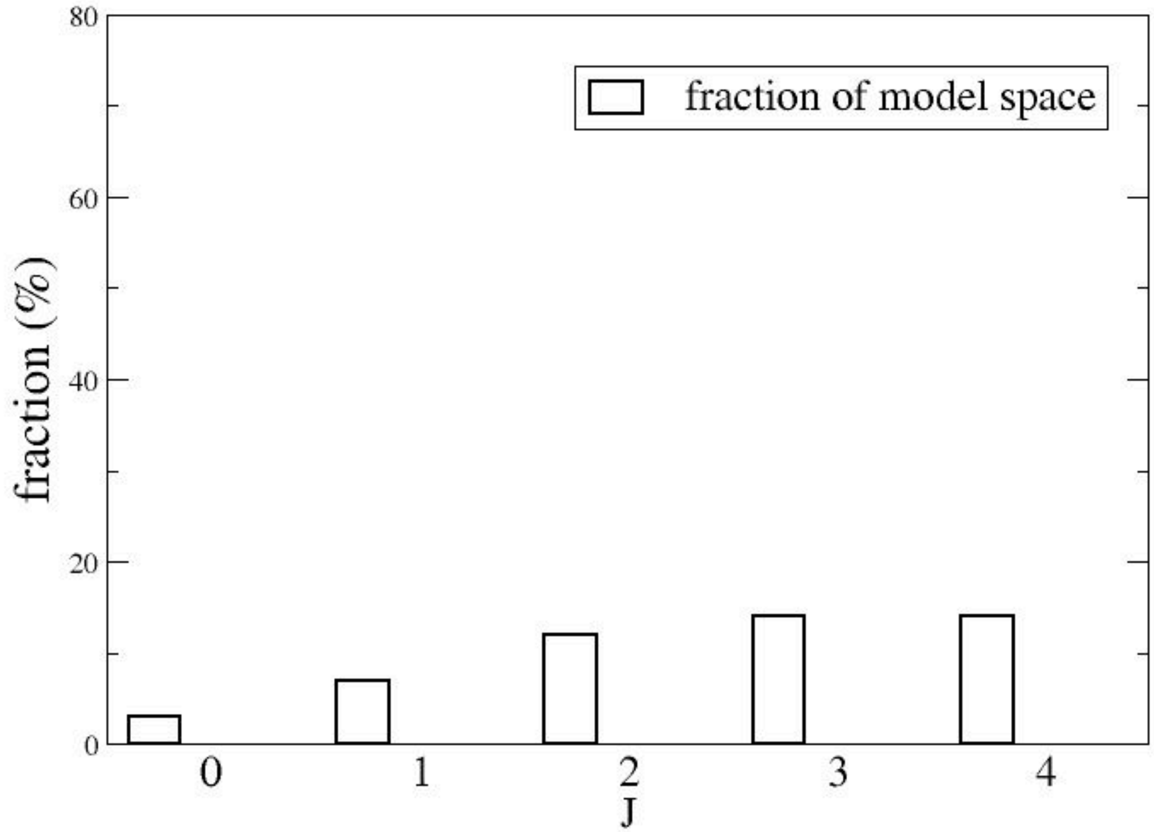
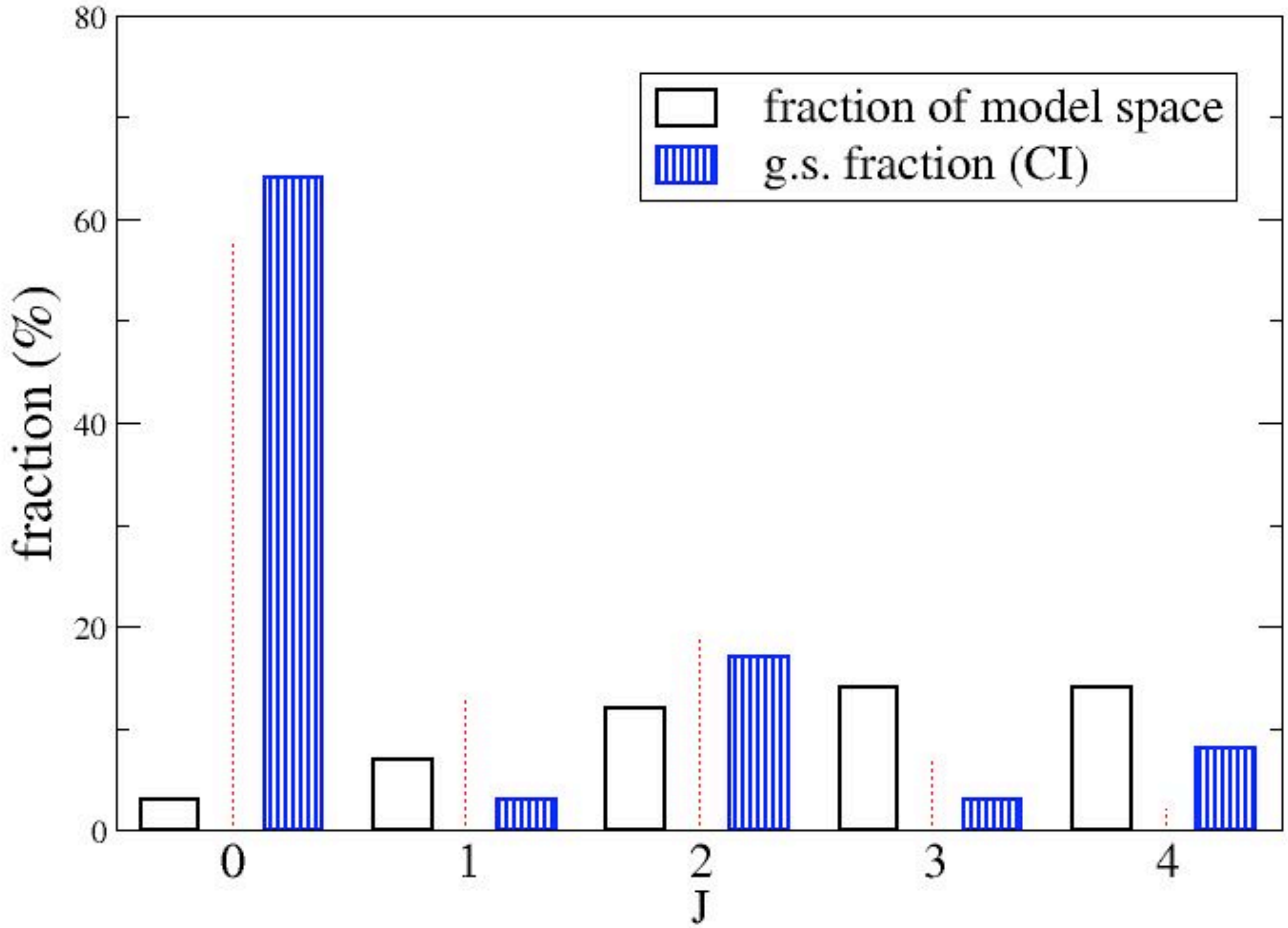
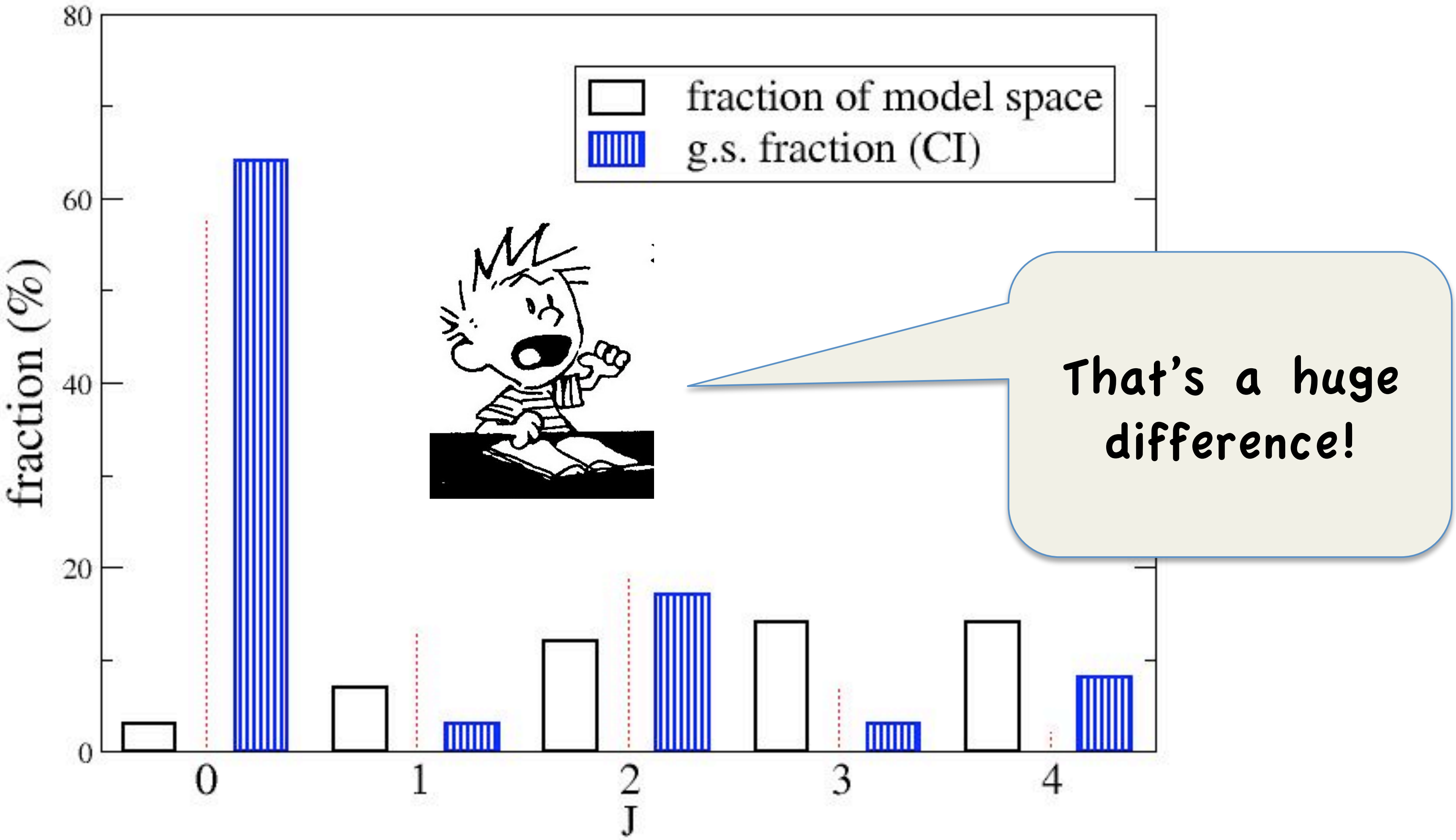


TABLE I. Percentage of ground states for selected random ensembles that have $J=0$ for our target nuclides, as compared to the percentage of all states in the model spaces that have these quantum numbers. (Statistical error is approximately $1-3\%$.) Entries with dashes were not computed.

Nucleus	$J=0$ (total space)	$J=2$ (total space)
	11.1%	14.8%
^{22}O	9.8%	13.4%
^{24}O	11.1%	14.8%
^{44}Ca	5.0%	9.6%
^{46}Ca	3.5%	8.1%
^{48}Ca	2.9%	7.6%
^{50}Ca	2.7%	7.1%
^{24}Mg	4%	16%
^{26}Mg	4%	15%
^{28}Mg	4%	16%



fraction of model space
g.s. fraction (CI)
fraction (%)
80
60
40
20
0
0
1
2
3
4
J


fraction of model space
g.s. fraction (CI)
fraction (%)
J
0
20
40
60
80
0
1
2
3
4
That's a huge difference!

Different ensembles of matrix elements

TABLE I. Percentage of ground states for selected random ensembles that have $J=0$ for our target nuclides, as compared to the percentage of all states in the model spaces that have these quantum numbers. (Statistical error is approximately $1-3\%$.) Entries with dashes were not computed.

Nucleus	RQE	RQE-NP	TBRE	RQE-SPE	$J=0$ (total space)	$J=2$ (total space)
^{20}O	68%	50%	50%	49%	11.1%	14.8%
^{22}O	72%	68%	71%	77%	9.8%	13.4%
^{24}O	66%	51%	55%	78%	11.1%	14.8%
^{44}Ca	70%	46%	41%	70%	5.0%	9.6%
^{46}Ca	76%	59%	56%	74%	3.5%	8.1%
^{48}Ca	72%	53%	58%	71%	2.9%	7.6%
^{50}Ca	65%	45%	51%	61%	2.7%	7.1%
^{24}Mg	66%	–	44%	54%	4%	16%
^{26}Mg	62%	52%	48%	56%	4%	15%
^{28}Mg	59%	46%	44%	54%	4%	16%

We have a long list of results that *qualitatively* resemble nuclear structure:

- •Pairing-like "gap" from g.s.
- •Odd-even staggering
- •One-particle, one-hole collectivity among low-lying states (band structure)

- •Mallman plots for J = 0,2,4,6,8 states

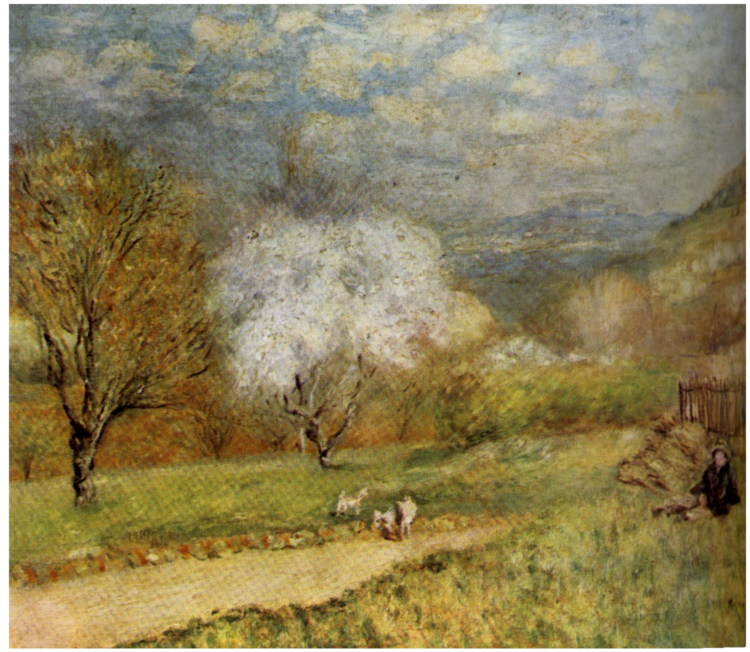
This is *amazing*!

Do we understand this?

“...the simple question of symmetry and chaos asks for a simple answer which is still missing.”

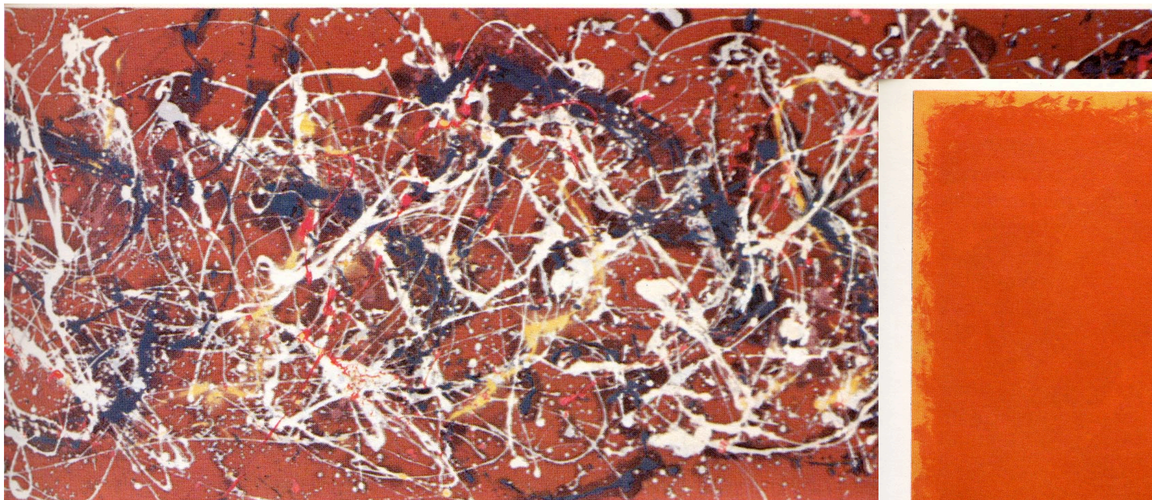
\- A. Volya, PRL **100**, 162501 (2008).

Bellini, *Madonna and Child*

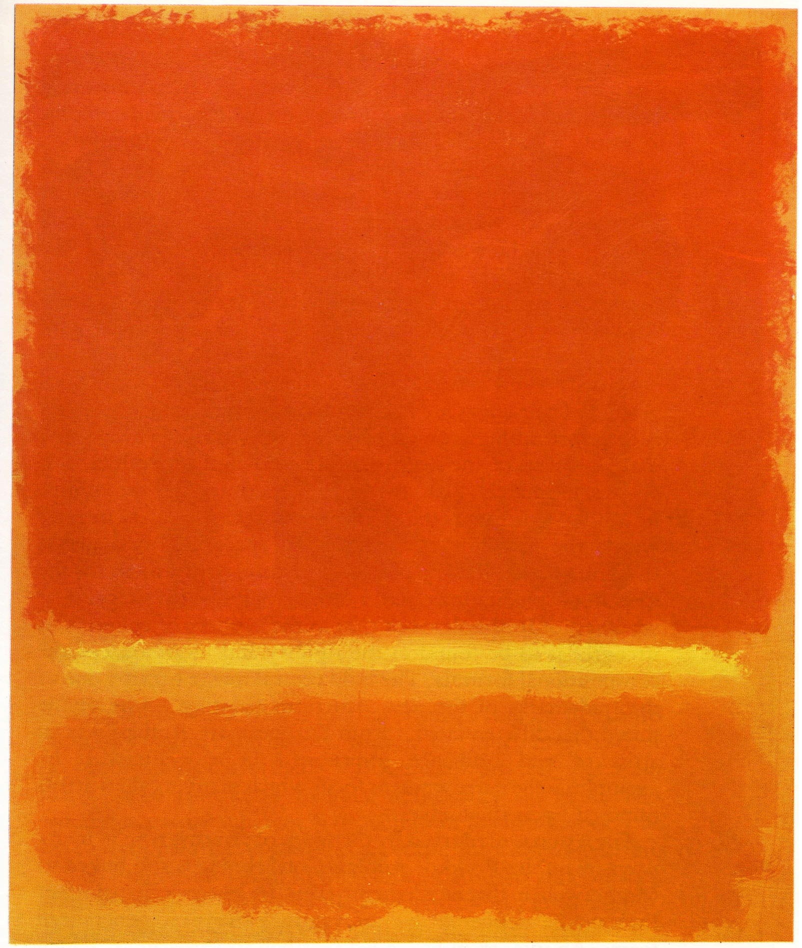


Renoir, *Country Road*

We’re not satisfied to *merely* represent reality…
in art (and science) we explore how far we can
stray and yet still “represent” some aspects



14 JACKSON POLLOCK *Number 2* 1949

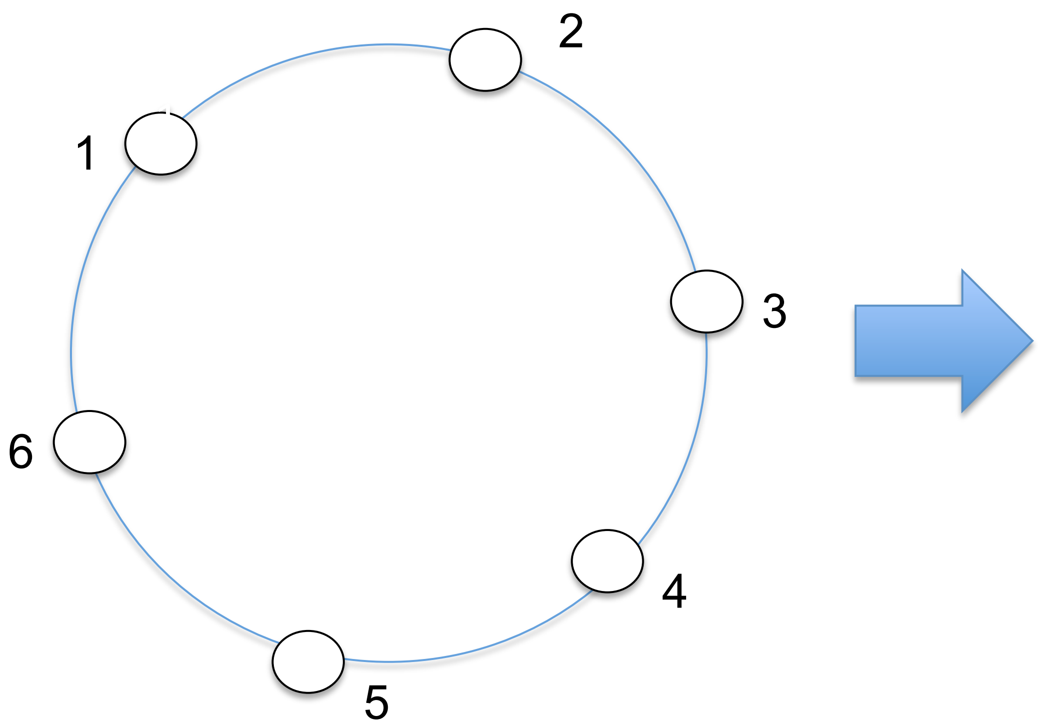


20 MARK ROTHKO *Orange Yellow Orange* 1969

Very simple systems may not seem realistic, but they probe the fundamentals in a way we can come to appreciate as beautiful

Can we go more abstract---
Can we impose a nontrivial symmetry on a random matrix*?

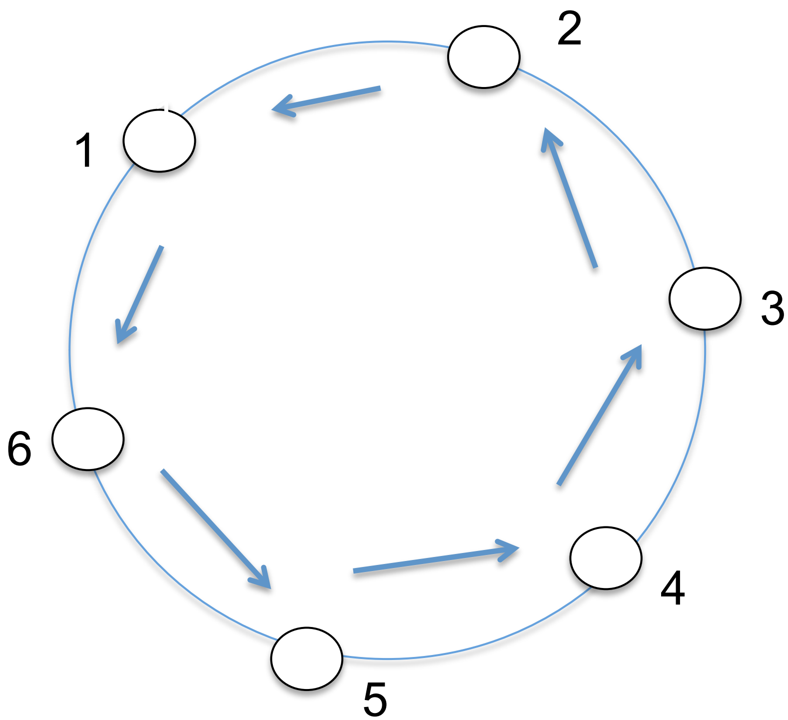
Consider C_n symmetry:



$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}$$

*e.g. Brody et al, RMP, 1981

The generator of rotations is



$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The generator of rotations is

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The general matrix invariant under $H = T^{-1} H T$ is

$$H = \begin{pmatrix} a & b & c & d & c & b \\ b & a & b & c & d & c \\ c & b & a & b & c & d \\ d & c & b & a & b & c \\ c & d & c & b & a & b \\ b & c & d & c & b & a \end{pmatrix}$$

Note that **H** is manifestly translationally invariant:

$$H_{ij} = F_{|i-j|}$$

$F_0 = a$, $F_1 = b$, $F_2 = c$, $F_3 = d$

The general matrix invariant under $H = T^{-1} H T$ is

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We can solve **H** by a Fourier transform; each eigenvalue is associated with a "quantum number" (momentum)

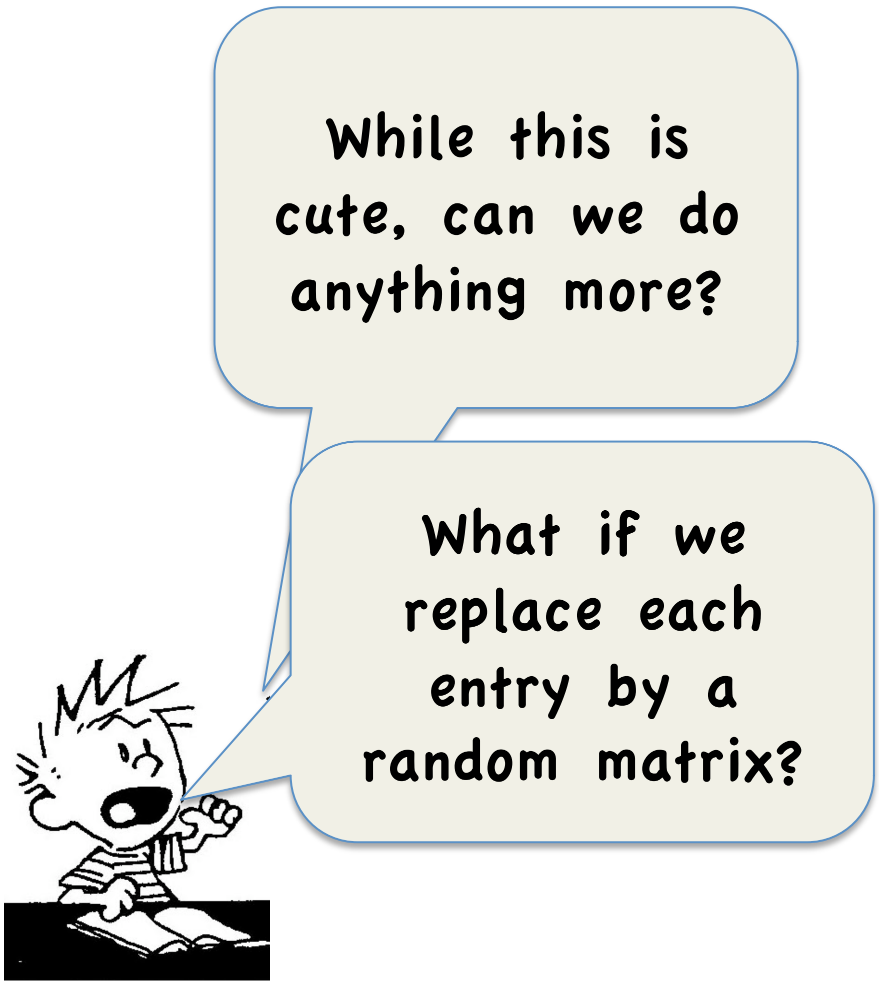
The general matrix invariant under H = T^{-1} H T is

$$h_m = \sum_k 2\cos\left(\frac{\pi m k}{N}\right) F_k$$

$$= \sum_k 2\cos\left(\frac{\pi m k}{N}\right) H_{1,1+k}$$

$$H = \begin{pmatrix} a & b & c & d & c & b \\ b & a & b & c & d & c \\ c & b & a & b & c & d \\ d & c & b & a & b & c \\ c & d & c & b & a & b \\ b & c & d & c & b & a \end{pmatrix}$$

(It's straightforward to also find the analytic eigenvectors—sines and cosines, as you'd imagine)



$$H = \begin{pmatrix} a & b & c & d & c & b \\ b & a & b & c & d & c \\ c & b & a & b & c & d \\ d & c & b & a & b & c \\ c & d & c & b & a & b \\ b & c & d & c & b & a \end{pmatrix}$$

$$H = \begin{pmatrix} A & B & C & D & C & B \\ B & A & B & C & D & C \\ C & B & A & B & C & D \\ D & C & B & A & B & C \\ C & D & C & B & A & B \\ B & C & D & C & B & A \end{pmatrix}$$

We can not longer analytically solve the matrix, *but* we *can* project out matrices representing the irreps (irreducible representations) of the symmetry:

As before, we identify the submatrices with an index:

$$\mathbf{F}_0 = \mathbf{A},\ \mathbf{F}_1 = \mathbf{B},\ \mathbf{F}_2 = \mathbf{C}\ldots$$

$$h_m = \sum_k 2\cos\left(\frac{\pi mk}{N}\right)F_k$$

$$H = \begin{pmatrix} A & B & C & D & C & B \\ B & A & B & C & D & C \\ C & B & A & B & C & D \\ D & C & B & A & B & C \\ C & D & C & B & A & B \\ B & C & D & C & B & A \end{pmatrix}$$

....only now $\mathbf{h}_m$ is a *matrix*.

Better yet, we can compute the *width* of each $\mathbf{h}_m$

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$$h_m = \sum_k 2\cos\left(\frac{\pi m k}{N}\right) F_k$$

....only now $\mathbf{h}_m$ is a *matrix*.

Transformed so **H′** is block-diagonal in irreps

$$H' = \begin{pmatrix} h_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_5 \end{pmatrix}$$

Better yet, we can compute the *width* of each $\mathbf{h}_m$

$$h_m = \sum_k 2\cos\left(\frac{\pi mk}{N}\right) F_k$$

Assuming all the submatrices are independent...

$$\sigma_m^2 = \sum_k 4\cos^2\left(\frac{\pi mk}{N}\right)\sigma^2(F_k)$$

Assuming all the submatrices have the *same width*...

$$\sigma_m^2 = \sum_k 4\cos^2\left(\frac{\pi mk}{N}\right)\sigma^2$$

$$\approx 2\sigma^2\left(1+\delta_{m,0}\right)$$

$$H = \begin{pmatrix} A & B & C & D & C & B \\ B & A & B & C & D & C \\ C & B & A & B & C & D \\ D & C & B & A & B & C \\ C & D & C & B & A & B \\ B & C & D & C & B & A \end{pmatrix}$$

$$H' = \begin{pmatrix} h_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_5 \end{pmatrix}$$

$$h_m = \sum_k 2\cos\left(\frac{\pi mk}{N}\right) F_k$$

Assuming all the submatrices are independent...

$$\sigma_m^2 = \sum_k 4\cos^2\left(\frac{\pi mk}{N}\right)\sigma^2(F_k)$$

Assuming all the submatrices have the same width...

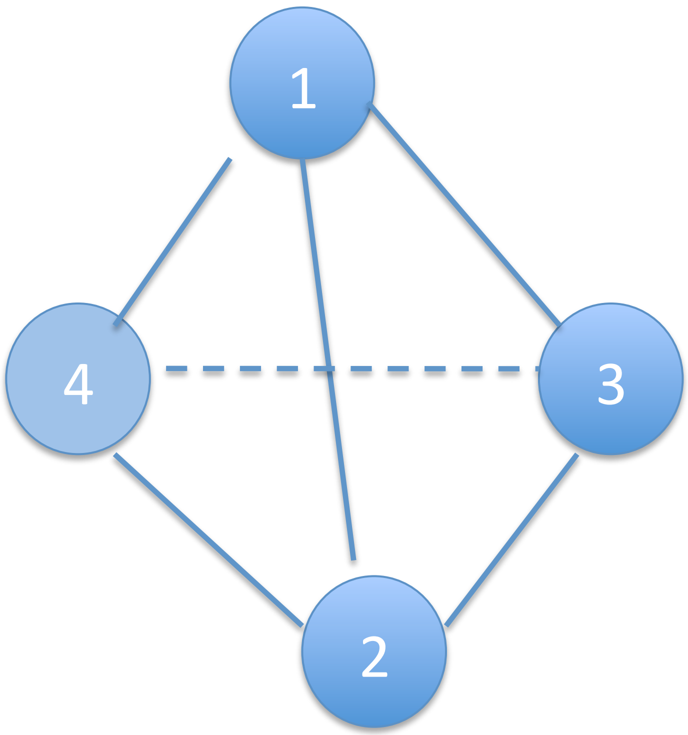
$$\sigma_m^2 = \sum_k 4\cos^2\left(\frac{\pi mk}{N}\right)\sigma^2$$

$$\approx 2\sigma^2\left(1+\delta_{m,0}\right)$$

What about other symmetries... particularly nonabelian symmetries?

Like the point groups?

The Tetrahedron



$$H = \begin{pmatrix} A & B & B & B \\ B & A & B & B \\ B & B & A & B \\ B & B & B & A \end{pmatrix}$$

One-dimensional irrep:
(most symmetric)

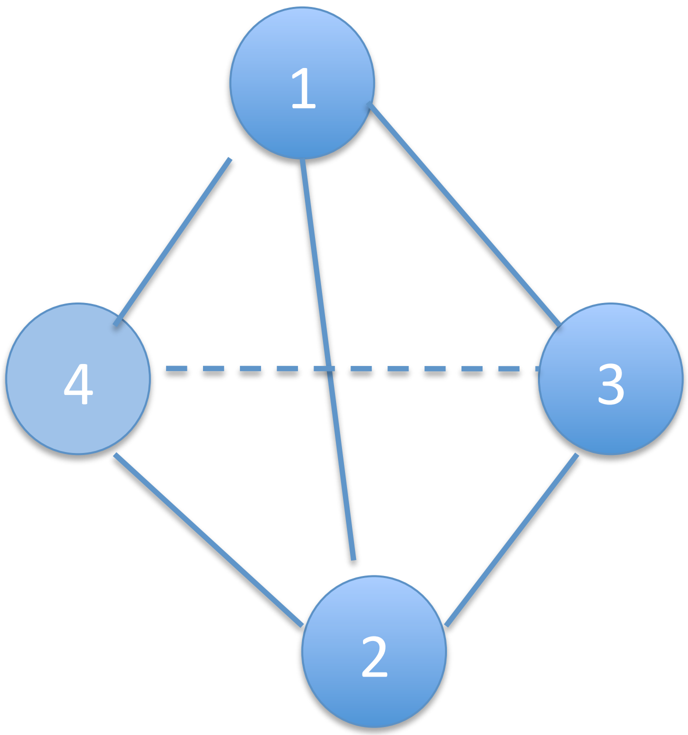
h= A+3B $\sigma^2{}_1 = 10$

Largest width so most likely ground state

3-dimensional irrep:

h = A – B $\sigma^2{}_3 = 2$

The Tetrahedron



Transformed so block-diagonal in irreps

$$H' = \begin{pmatrix} A+3B & 0 & 0 & 0 \\ 0 & A-B & 0 & 0 \\ 0 & 0 & A-B & 0 \\ 0 & 0 & 0 & A-B \end{pmatrix}$$

One-dimensional irrep:
(most symmetric)

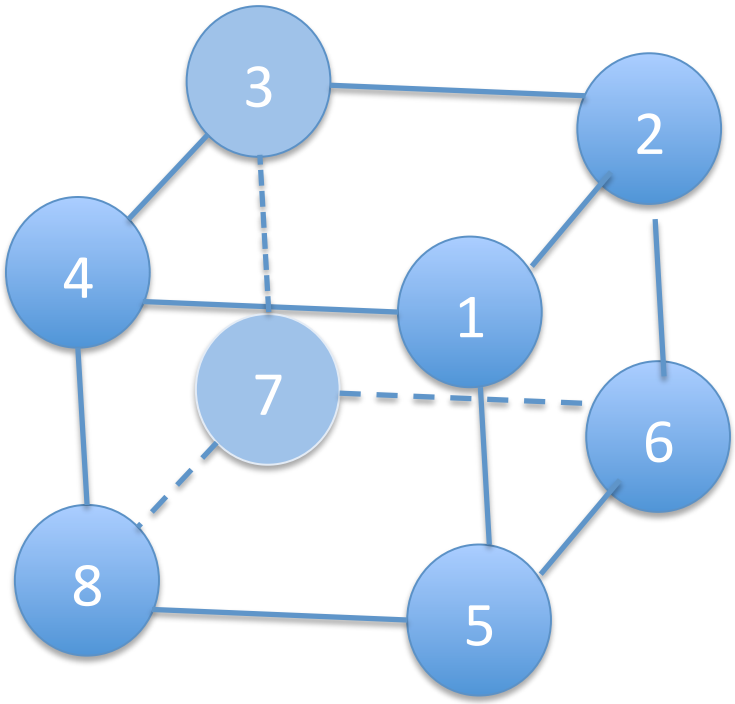
h= A+3B $\sigma^2{}_1 = 10$

Largest width
so most likely
ground state

3-dimensional irrep:

h = A – B $\sigma^2{}_3 = 2$

The Cube



$$H = \begin{pmatrix} A & B & C & B & B & C & D & C \\ B & A & B & C & C & B & C & D \\ C & B & A & B & D & C & B & C \\ B & C & B & A & C & D & C & B \\ B & C & D & C & A & B & C & B \\ C & B & C & D & B & A & B & C \\ D & C & B & C & C & B & A & B \\ C & D & C & B & B & C & B & A \end{pmatrix}$$

One-dimensional irreps:
(most symmetric)

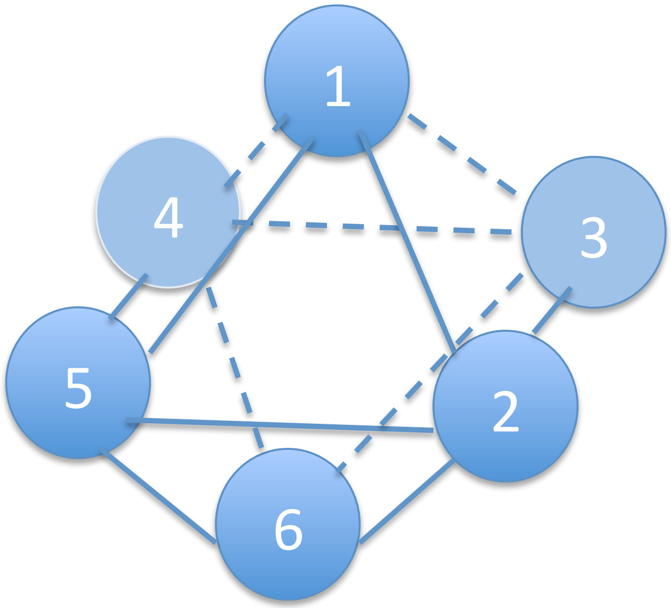
h= A±3B +3C±D $\sigma^2{}_1 = 20$

Largest width so most likely ground state

3-dimensional irreps:

h = A-C ± (B-D) $\sigma^2{}_3 = 4$

The Octahedron



$$H = \begin{pmatrix} A & B & C & B & B & B \\ B & A & B & C & B & B \\ C & B & A & B & B & B \\ B & C & B & A & B & B \\ B & B & B & B & A & C \\ B & B & B & B & C & A \end{pmatrix}$$

One-dimensional irrep:
(most symmetric)

h= A+4B +C $\sigma^2{}_1 = 18$

Largest width so most likely ground state

2-dimensional irrep:

h = A-2B+C $\sigma^2{}_2 = 6$

3-dimensional irrep:

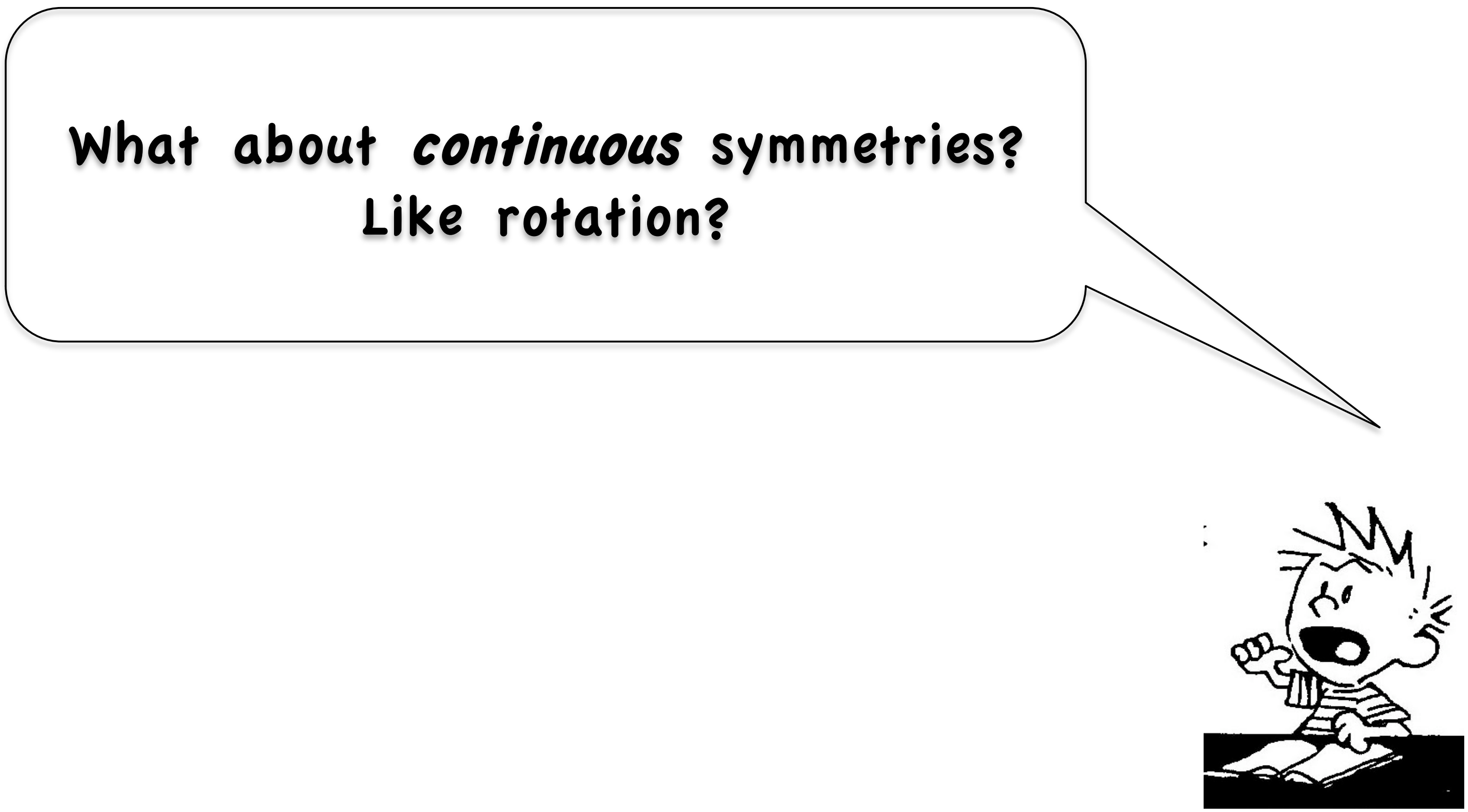
h = A-C $\sigma^2{}_3 = 2$

What have we learned so far?

If we impose symmetries on a random matrix (leaving additional degrees of freedom)....

... the lowest dimension / "most symmetric" irreps have largest widths and thus dominate the ground state

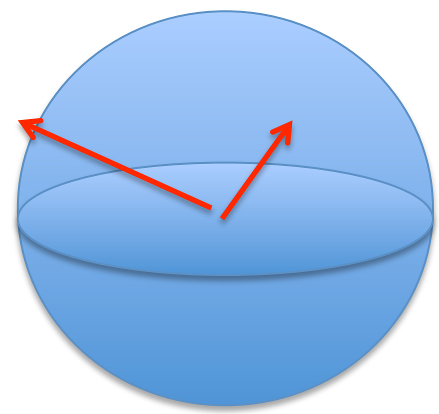
Happy Birthday Franco!



Starting from a rotationally invariant Hamiltonian:

$$H\left(\theta'\phi',\theta\phi\right) = F(\omega)$$

$$\cos\omega = \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos(\phi' - \phi)$$



....we can project out Hamiltonians with good *L*:

$$H_L = 2\pi\int_0^\pi P_L(\cos\omega)F(\omega)d\cos\omega$$

From this we can compute the width as a function of *L*:

$$\sigma_L^2 = 4\pi^2\int_0^\pi P_L^2(\cos\omega)\sin^2\omega d\omega$$

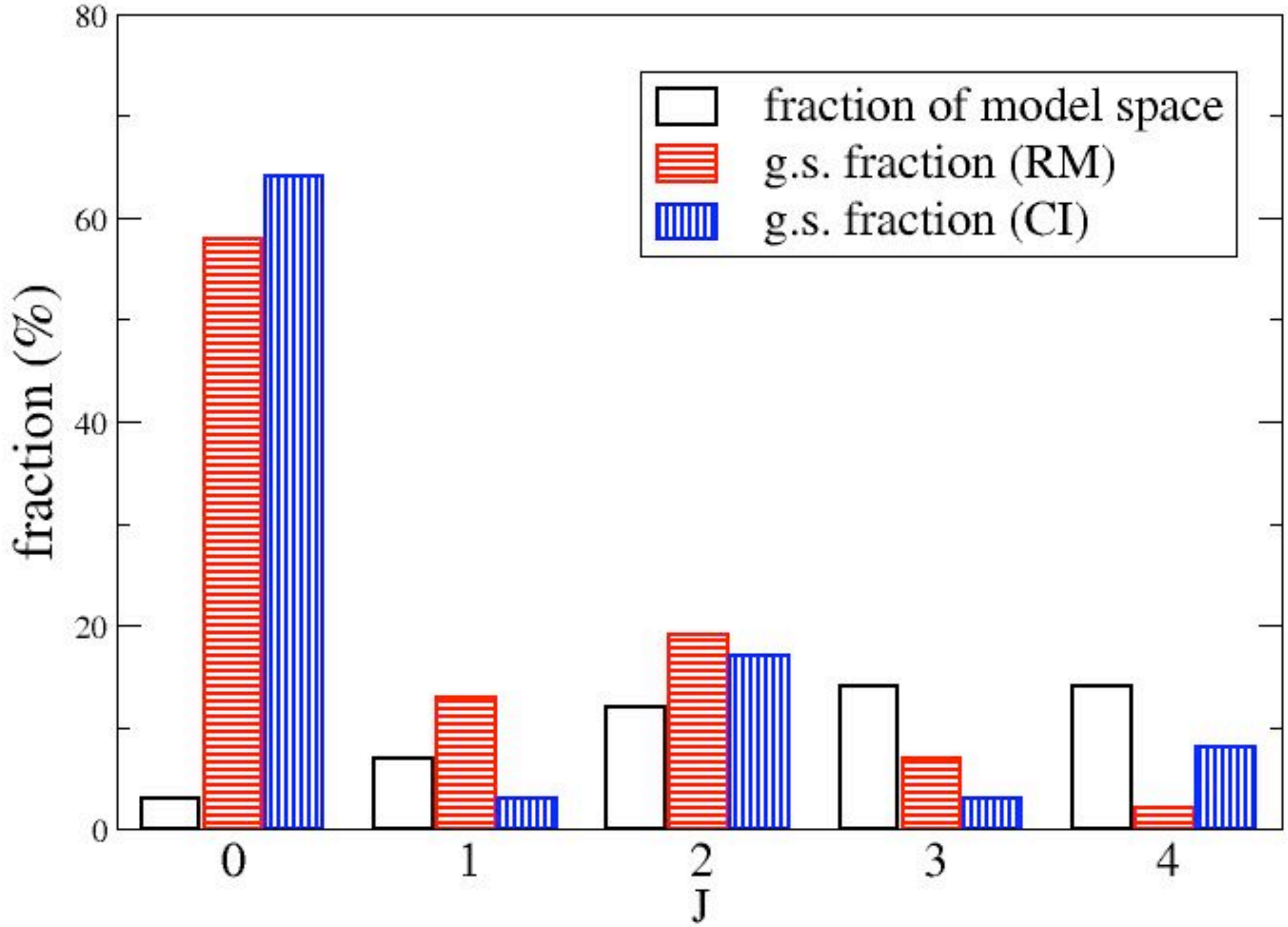
For *L*= 0, 1,2,3, 4 values: 1.571, 0.393, 0.245, 0.178, 0.139

Mapping onto many-body simulations is not trivial:

-- Different J spaces have different dimensions

-- Level densities is Gaussian, not GOE

To account for this, choose Gaussian with width $$\sigma_L(eff) = \sqrt{N_L}\sigma_L$$


80
60
40
20
0
fraction (%)
0
1
2
3
4
J
fraction of model space
g.s. fraction (RM)
g.s. fraction (CI)

“single-*j* shell: $(21/2)^8$

J	f_{space} (%)	f_{RM}	f_{CI} (%)
0	0.4	33	55
1	0.5	0.2	0
2	1	9	7
3	1	3	0.2
4	2	11	2

IBM, N = 7

J	f_{space} (%)	f_{RM}	f_{CI} (%)
0	11	81	55
1	N/A	-	-
2	17	14	13
3	6	0.1	0.08
4	17	4	4

For even *more* results, come to HITES (a.k.a. Draayerfest) in 3 weeks...

Summary:

Symmetries lead to conserved quantities (E. Noether)
= “quantum numbers”

By considering random matrices with symmetries,
we find that the ground state is dominated by
lowest-dimension / most symmetric irrep

...a “beautiful” results

The basic question here:

How much *choice* is there in dynamical systems?
(Einstein: “What really interests me is whether God had any choice in the creation of the world.”)

i.e., having a J=0 g.s. doesn’t tell us much about the interaction...

...but some other features are likely more diagnostic

Work to be done:

Need to formalize results from points groups.

Make application to continuous symmetries more rigorous.

Can I better motivate mapping/modeling of many-body systems?

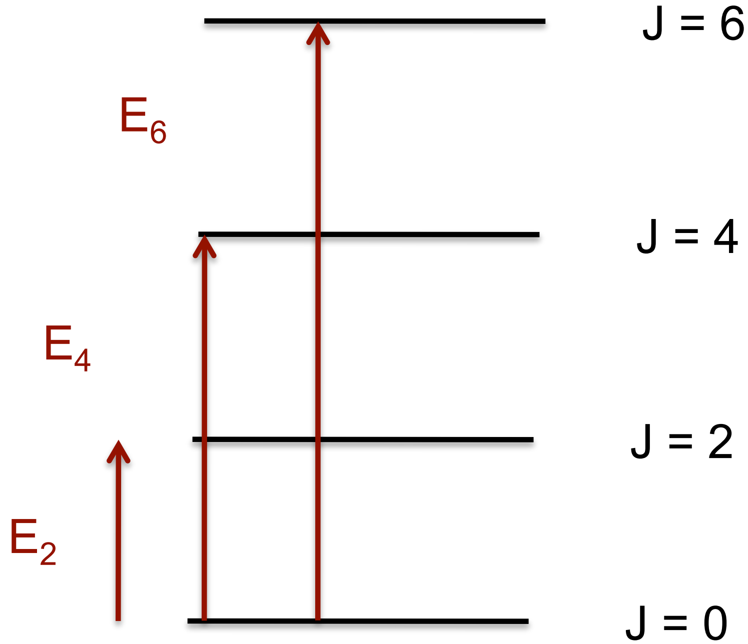
Symmetry breaking and partial/quasi-dynamical symmetry

What about other phenomena? Such as R_{62}/ R_{42} ratio?
(Preliminary results suggest a strong correlation; furthermore the equivalent R_{12}/R_{42} has no correlation – a prediction!)

Happy Birthday Franco!

A lot of fun work ahead!

"Beauty in Physics" Cocoyoc May 2012



Seniority:

$R_{42} = R_{62} = 1$

Vibrational:

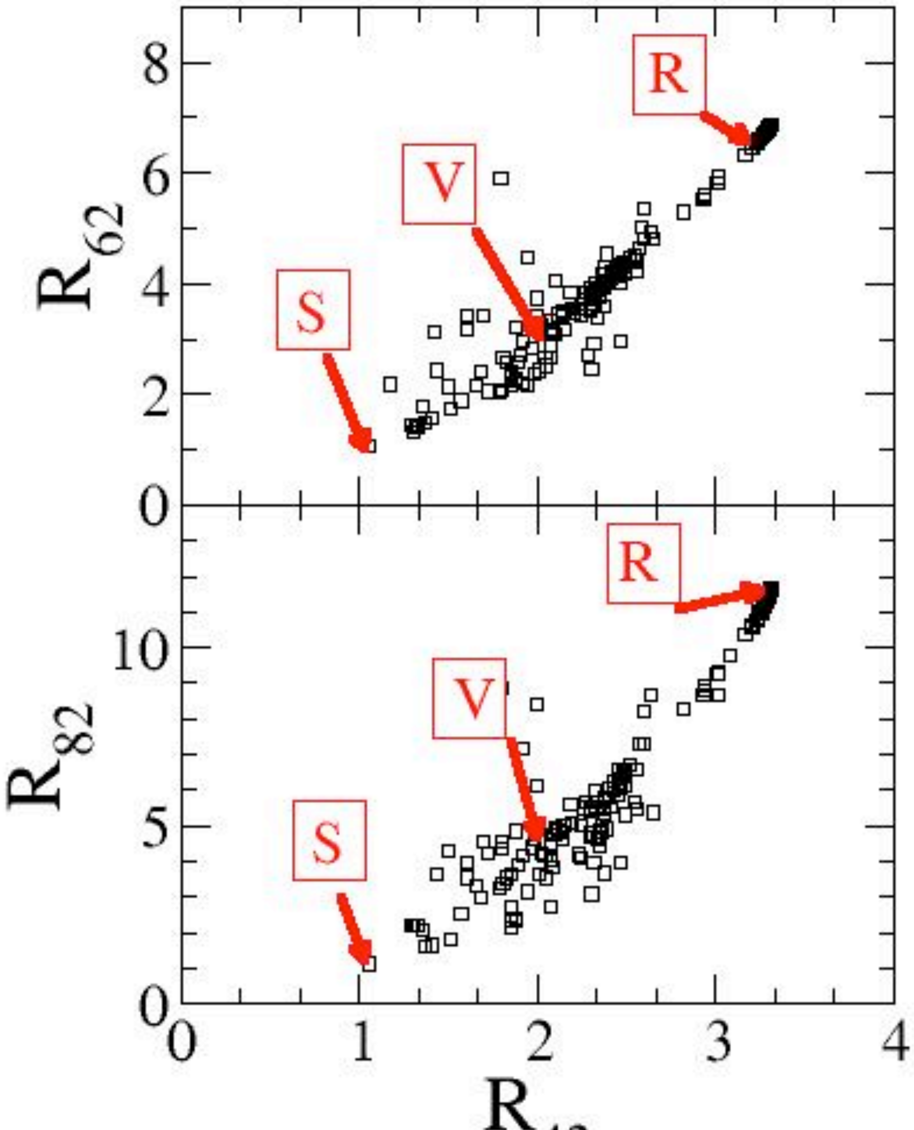
$R_{42} = 2$, $R_{62} = 3$

Rotational:

$R_{42} = 3.33$, $R_{62} = 7$

$$R_{42} = E_4 / E_2$$

$$R_{62} = E_6 / E_2$$

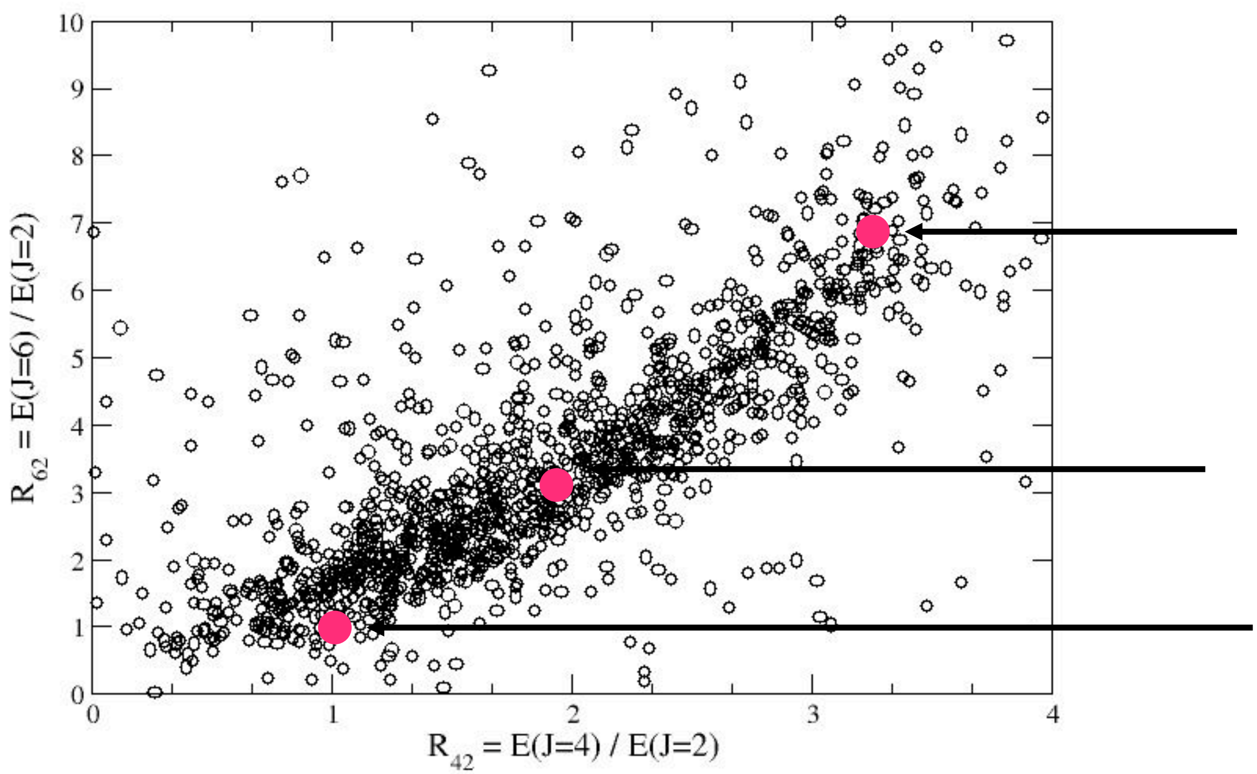


Data taken from *all* stable even-even nuclides

Almost a one-parameter family!

Plot E(6)/E(2) vs E(4)/E(2)

“^{24}Mg”



rotational limit

vibrational limit

“seniority” limit

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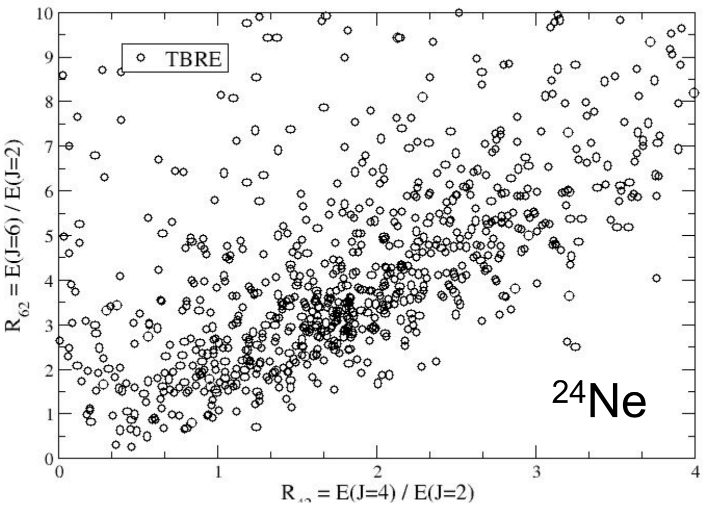
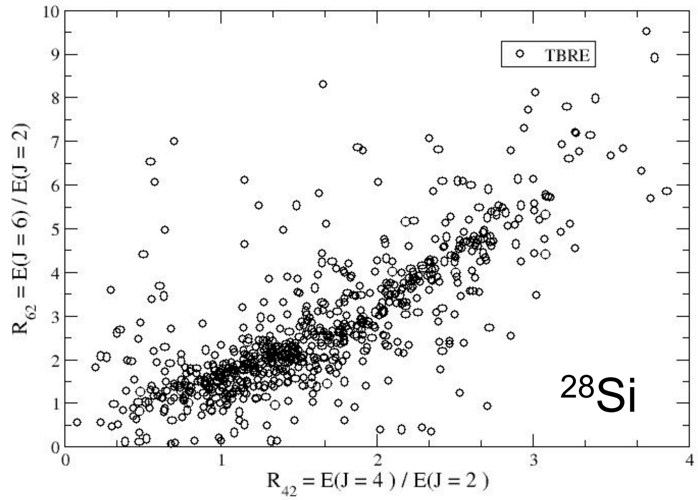
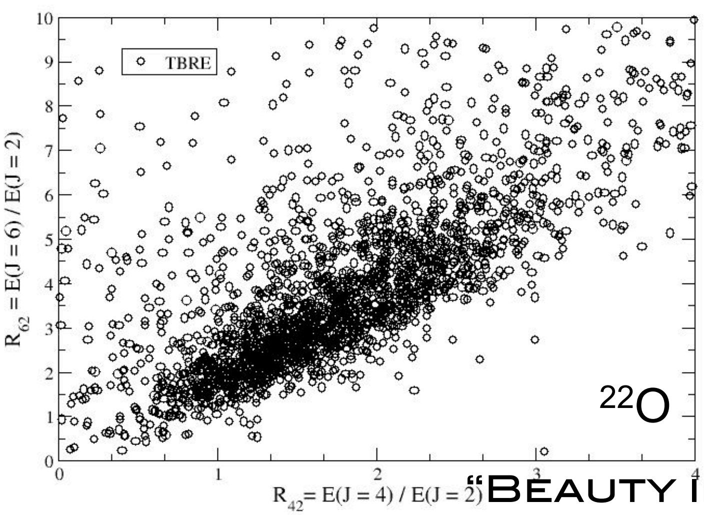
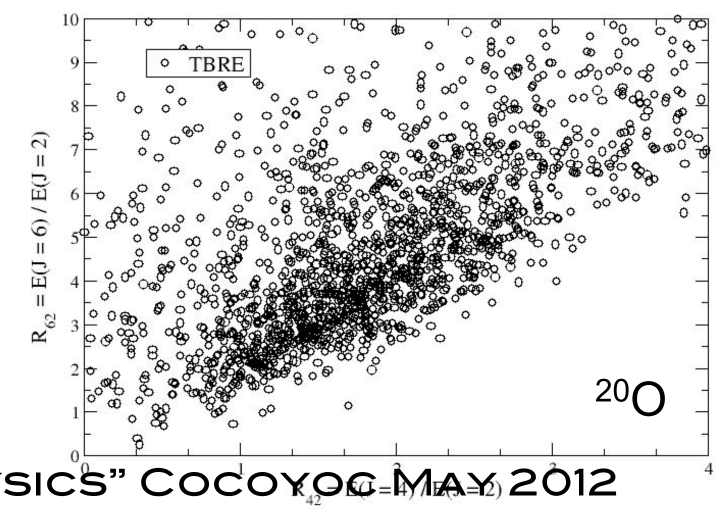
Plot E(6)/E(2) vs E(4)/E(2)









Interacting Boson Model (IBM)

J = 6

E_6/E_2 vs E_4/E_2

J = 8

E_8/E_2 vs E_4/E_2