Matter-Field Entanglement within the Dicke Model

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The Dicke Hamiltonian

The Dicke model describes a cloud of cold atoms interacting with a one-mode electromagnetic field in an optical cavity [Dicke 1954]

$$H_D = \hat{a}^\dagger \hat{a} + \omega_A \hat{J}_z + \frac{\gamma}{\sqrt{N}} \left(\hat{a}^\dagger + \hat{a}\right) \left(\hat{J}_+ + \hat{J}_-\right) \ .$$

Consider only the symmetric configuration of atoms i.e., $N = 2j$.

The model presents a quantum phase transition when the coupling constant, for $N \to \infty$, takes the value $\gamma_c = \sqrt{\omega_A}/2$ [Hepp, Lieb 1973].

Energy surface for CS

The energy surface, or associated classical Hamilton function, is formed by means of the expectation value of the Hamiltonian with respect to a test state.

We consider as test state the direct product of coherent states of the groups $HW(1)$, for the electromagnetic part (Glauber 1963) and $SU(2)$, for the atomic part (Arecchi et al 1972)

, i.e.,

$$|\alpha\rangle\otimes|\zeta\rangle = \frac{e^{-|\alpha|^2/2}}{\left(1+|\zeta|^2\right)^j}\sum_{\nu=0}^{\infty}\sum_{m=-j}^{+j}\left\{\frac{\alpha^\nu}{\sqrt{\nu!}}\binom{2j}{j+m}^{1/2}\zeta^{j+m}|\nu\rangle\otimes|j,\,m\rangle\right\}.$$

Energy surface for CS

The energy surface is given by

$$\begin{aligned}\mathcal{H}(\alpha,\,\zeta) &\equiv \langle\alpha|\otimes\langle\zeta|H_D|\alpha\rangle\otimes|\zeta\rangle \\ &= \frac{1}{2}\left(p^2+q^2\right)-j\,\omega_A\,\cos\theta+2\sqrt{j\gamma}\,q\,\sin\theta\,\cos\phi\ .\end{aligned}$$

We define

$$\begin{aligned}\alpha &= \frac{1}{\sqrt{2}}\,(q+i\,p)\ , \\ \zeta &= \tan\left(\frac{\theta}{2}\right)\exp\left(i\,\phi\right)\ ,\end{aligned}$$

where $(q,\,p)$ correspond to the expectation values of the quadratures of the electromagnetic field and $(\theta,\,\phi)$ determine a point on the Bloch sphere.

Critical points of the CS Energy surface

The critical points that minimize the energy surface, for $\omega_A > 0$,

$$\theta_c = 0,\ q_c = 0,\ p_c = 0,$$
$$|\gamma| < \gamma_c\ ;$$
$$\theta_c = \arccos(\gamma_c/\gamma)^2,\ q_c = -2\sqrt{j}\,\gamma\sqrt{1-(\gamma_c/\gamma)^4}\cos\phi_c,$$
$$p_c = 0,\ \phi_c = 0,\ \pi,$$
$$|\gamma| > \gamma_c\ .$$

$\gamma_c = \sqrt{\omega_A}/2$. From the expressions for the critical values in the superradiant regime, where $\theta_c \neq 0$, one gets

$$\frac{q_c}{\sqrt{N}} = -\sqrt{\omega_A}\frac{\sin\theta_c}{\sqrt{2\cos\theta_c}}\cos\phi_c\ .$$

Symmetry Adapted Coherent States

The test states (CS) do not respect the symmetry of the Dicke Hamiltonian. Then it is convenient to define states adapted to the discrete symmetry present in H_D,

$$\begin{aligned} |\alpha, \zeta\rangle_{\pm} &= \mathcal{N}_{\pm} \frac{1}{2}\left(1 \pm \exp(i\pi\hat{\Lambda})\right) |\alpha, \zeta\rangle \\ &= \mathcal{N}_{\pm} \left(|\alpha\rangle \otimes |\zeta\rangle \pm |-\alpha\rangle \otimes |-\zeta\rangle\right) . \end{aligned}$$

where $\hat{\Lambda} = j + \hat{N}_{ph} + \hat{J}_z$ and the normalization

$$\mathcal{N}_{\pm} = 1 \Big/ \sqrt{2\left(1 \pm \exp\left(-2|\alpha|^2\right)\left(\frac{1-|\zeta|^2}{1+|\zeta|^2}\right)^N\right)} .$$

Energy surface for SAS

The energy surface for the symmetry adapted states is given by

$$\langle H_D \rangle_{\pm} = \pm \frac{1}{2}\left(p^2+q^2\right)\left\{1-\frac{2}{1\pm e^{\pm(p^2+q^2)}(\cos\theta)^{\mp N}}\right\}$$
$$-\frac{N}{2}\,\omega_A\left\{(\cos\theta)^{\pm 1}\pm\frac{\tan^2\theta\,\cos\theta}{1\pm e^{\pm(p^2+q^2)}(\cos\theta)^{\mp N}}\right\}$$
$$+\sqrt{2N}\,\gamma\left\{\frac{\pm p\,\tan\theta\,\sin\phi+q\,e^{p^2+q^2}\,\sin\theta\,\cos\phi\,(\cos\theta)^{-N}}{e^{p^2+q^2}(\cos\theta)^{-N}\pm 1}\right\}\,.$$

Energy surface for SAS

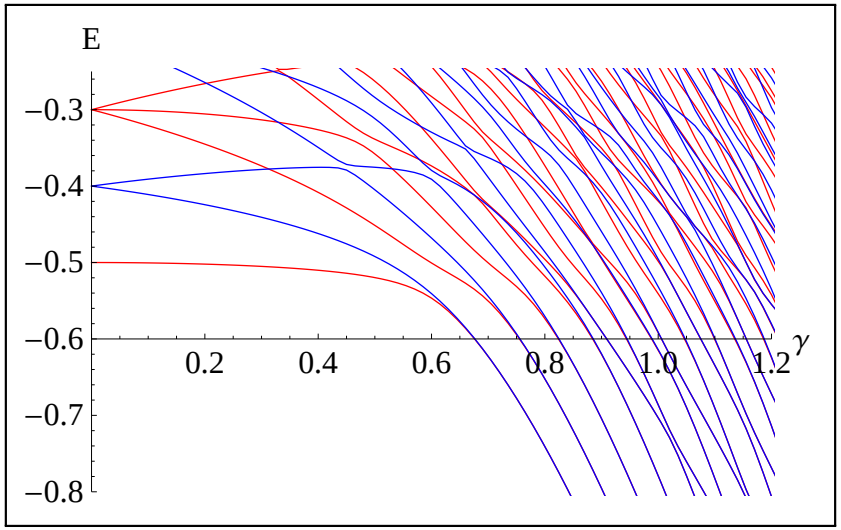
There are two procedures, one substitutes the analytical critical points for the CS in the energy surface for the SAS; the same procedure can be used to determine the expectation values of other observables of interest (OC et al Phys. Rev A 84 (2011)).

The second one is the following: For a given number of atoms N together with a fix value of the coupling parameter γ, we determine the values q_c, p_c, θ_c, ϕ_c, where the energy surface presents a minimum.

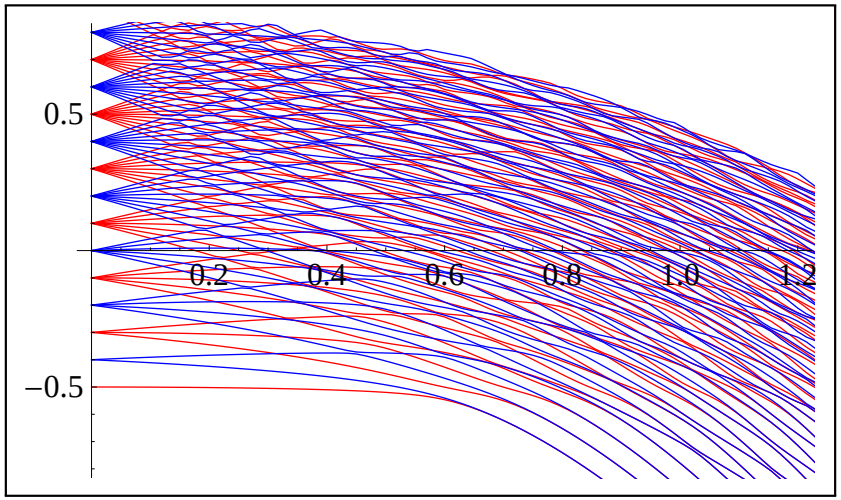
The minima always occur for $p_c = 0$ and $\phi_c = 0, \pi$. In this presentation I will show you results for $\phi_c = 0$.

The values of q_c and θ_c have to be determine numerically.

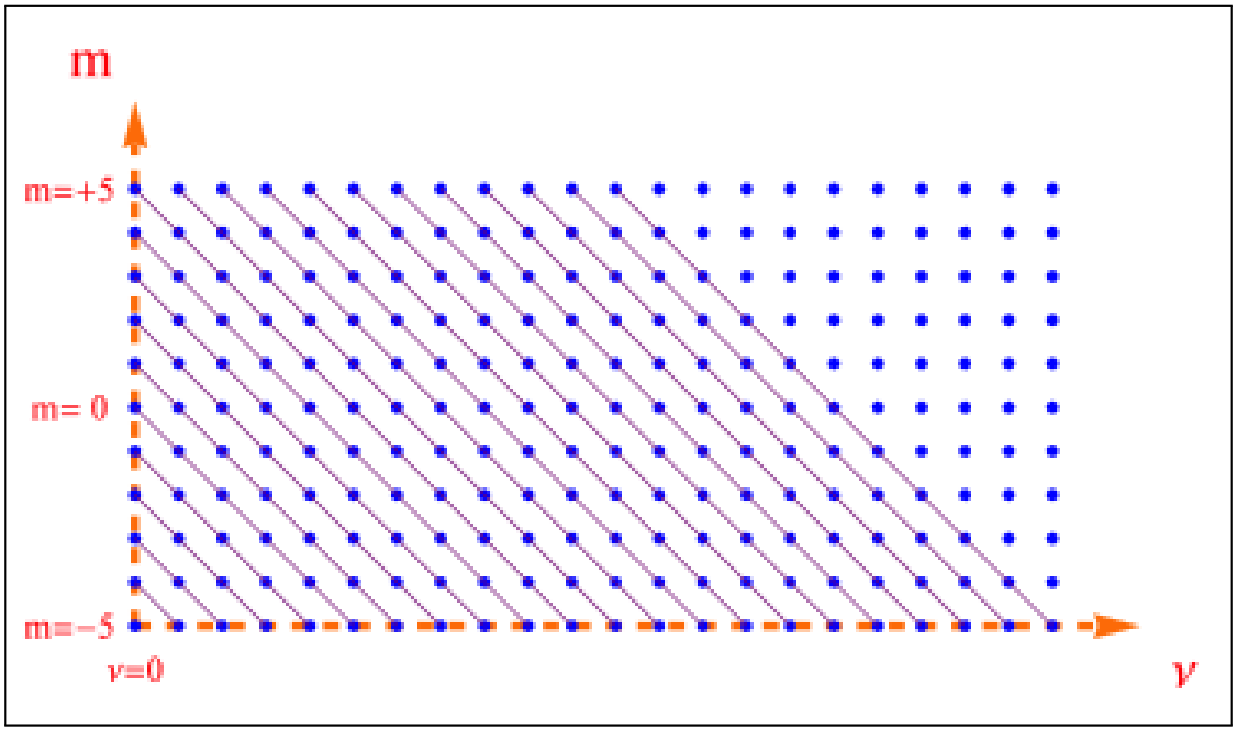
Energy spectra



Energy spectra of H_D, for $N = 10$ atoms. The size of the space is fixed to get convergence in the values of the energy spectra.

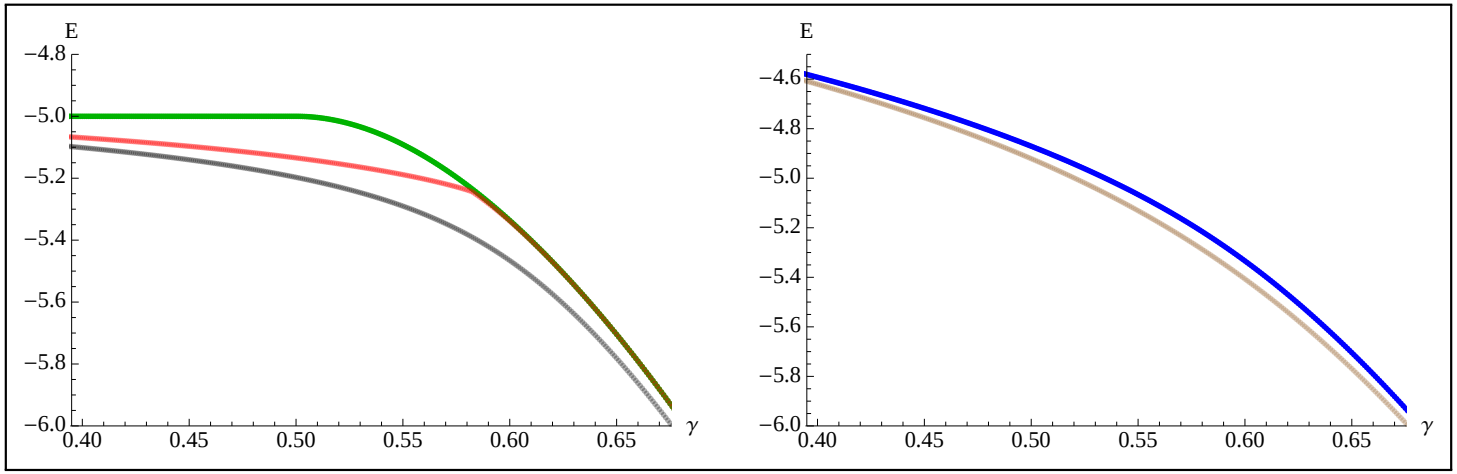


Quantum states

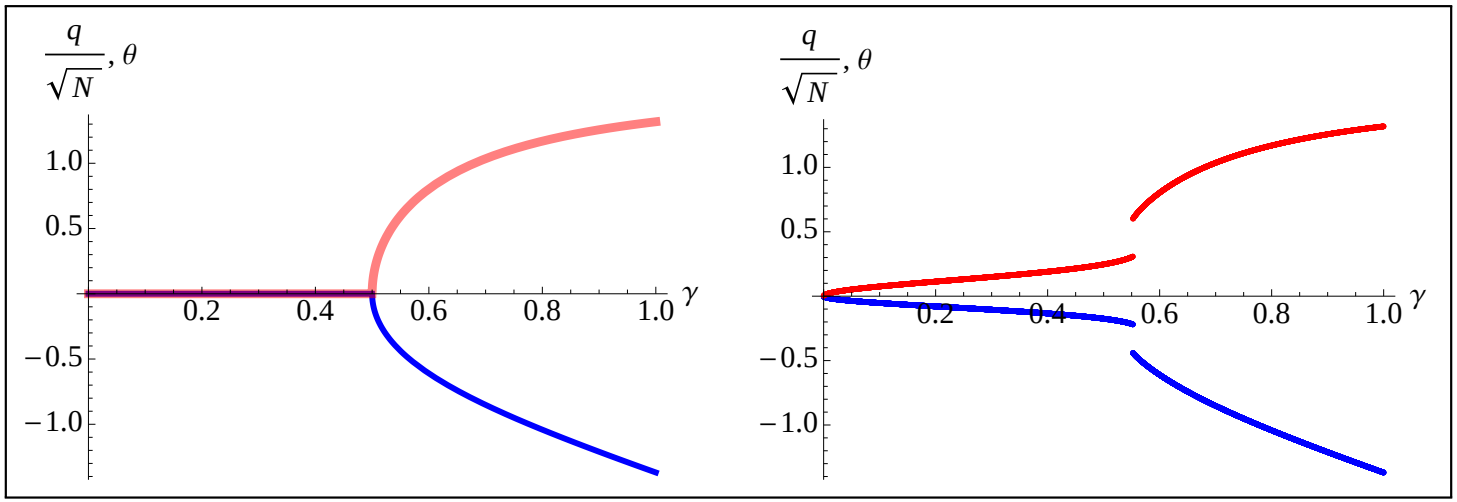


The basis states can be written

$$|\psi_k\rangle = \sum_{\lambda=0}^{\infty} \sum_{\nu=\max(0,\lambda-2j)}^{\lambda} c_k(\lambda,\nu)|\nu\rangle|j,\lambda-j-\nu\rangle\,.$$

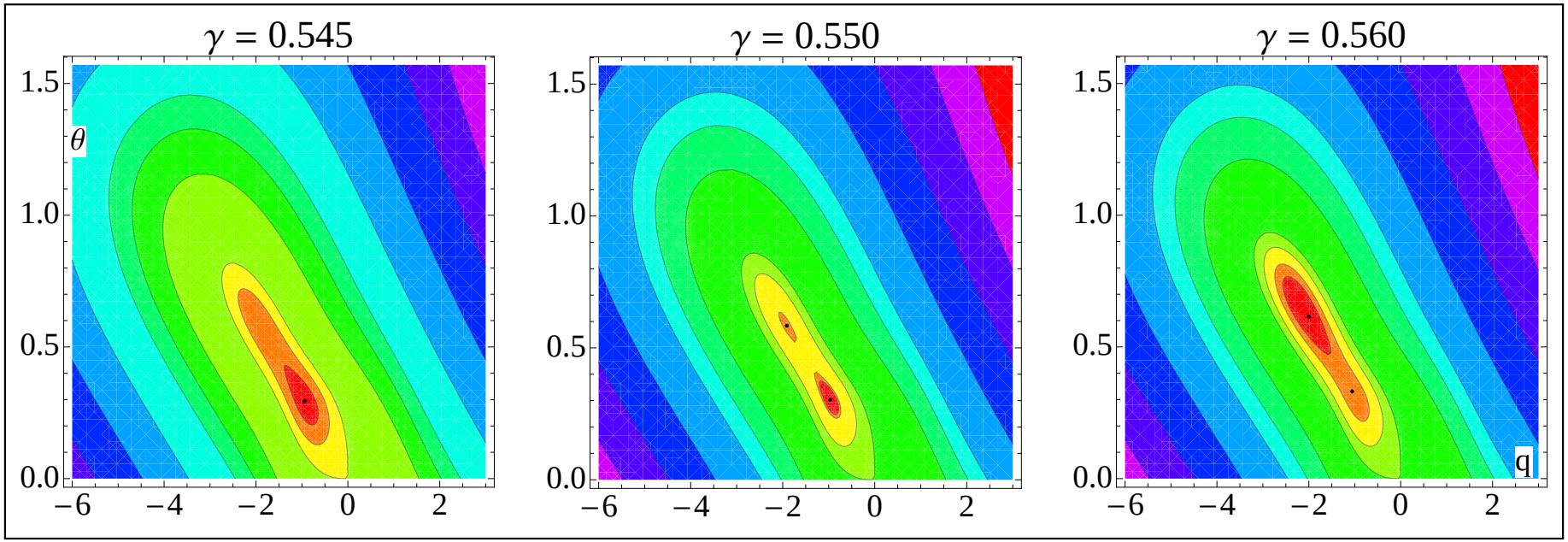


At the left the ground state energy of exact (gray), SAS (red), CS (green), while at the right the first excited energies of exact (brown), SAS (blue) as functions of γ, for $\omega_A = 1$ and $N = 20$ atoms.



The phase transitions can be seen clearly in the expressions for q and θ.

Contours Energy plots for SAS

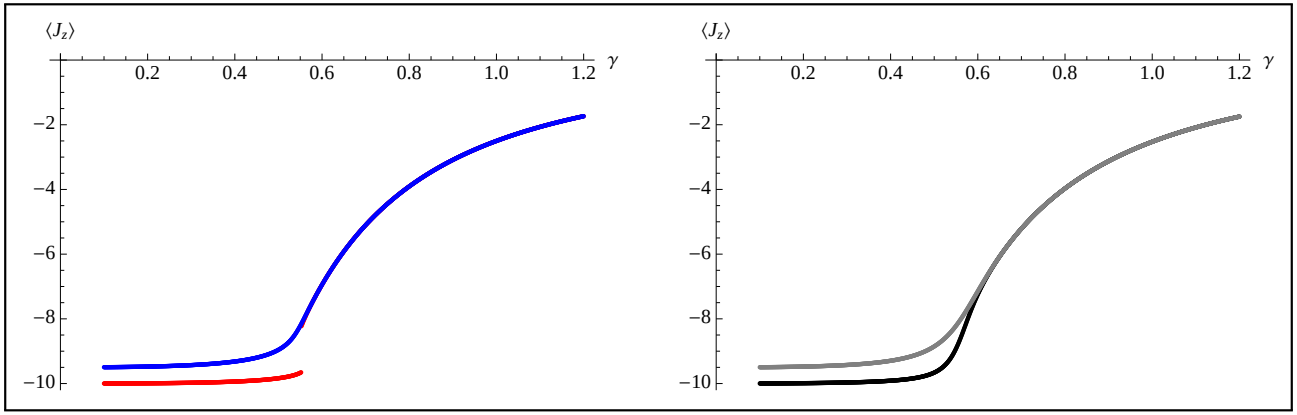


We take $N = 20$ atoms and $\omega_A = 1$.

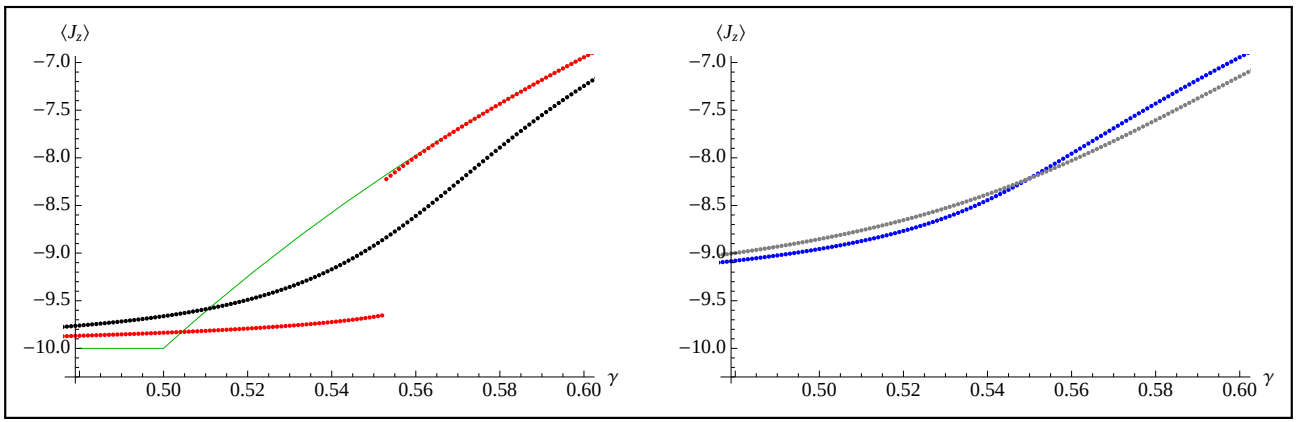
The values $\{E_{min}, q_{min}, \theta_{min}\}$ are from left to right, $\{-10.1887, -0.935972, 0.29446\}$, $\{-10.1963, -0.964931, 0.30329\}$ y $\{-10.2137, -1.05681, 0.331208\}$, respectively.

The plot shows why there is a jump in the minimum critical points.

Comparison Variational vs Exact Results

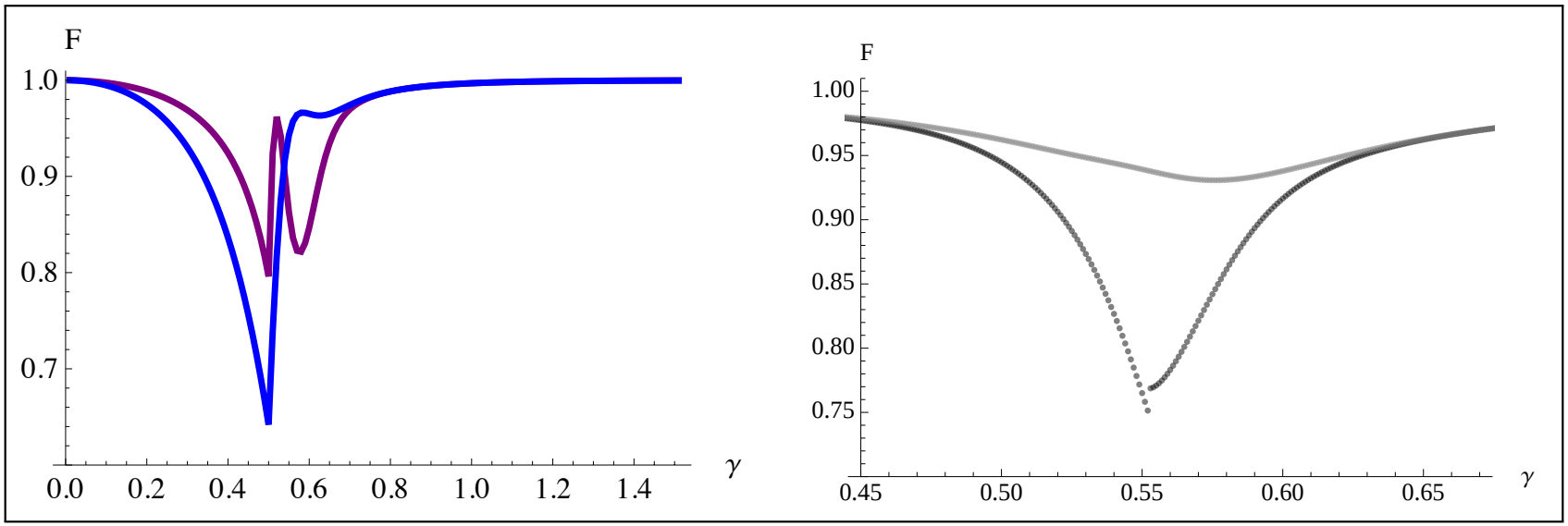


Quantum, SAS, and CS results for $\langle J_z \rangle$.



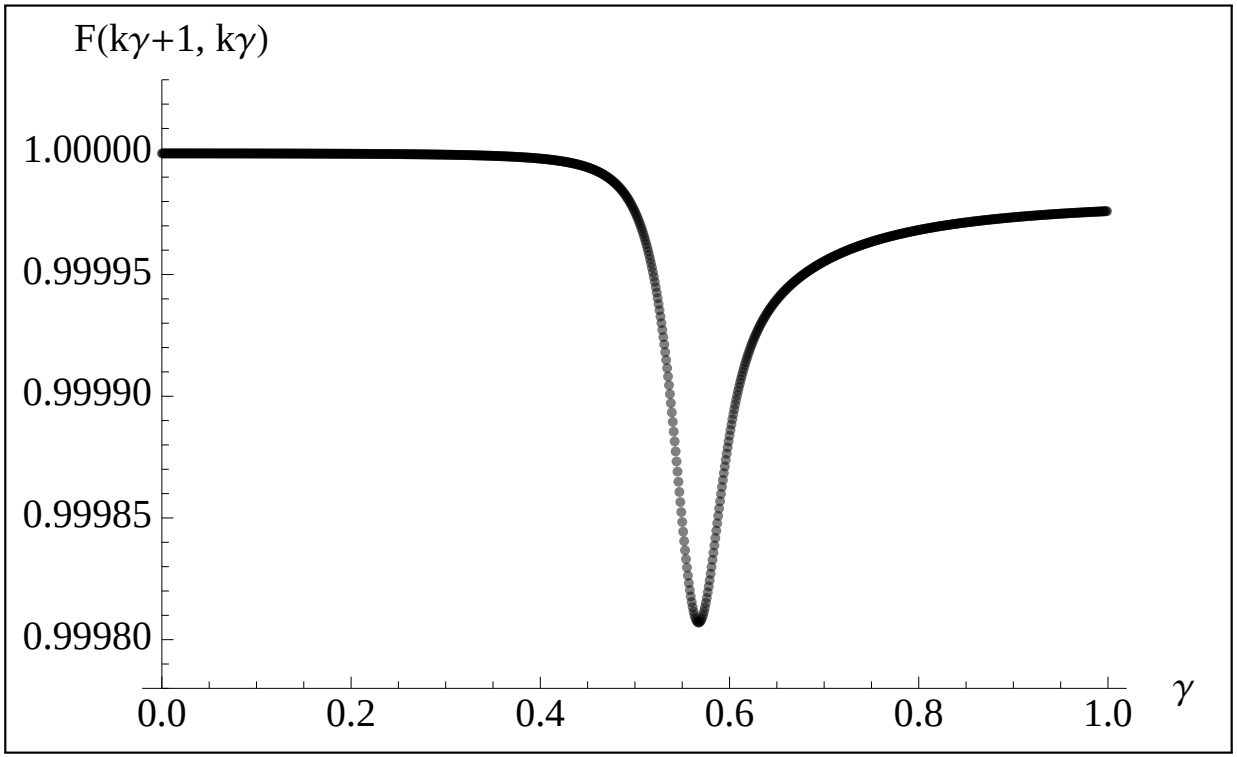
Comparison between the variational CS, SAS and EXACT results for the even case. The comparison between SAS and EXACT results for the odd case.

Fidelity



The fidelity is a measure to establish how similar is a probability distribution to another one. When the system is pure this reduces to the overlap between the corresponding states. At the left, the fidelity between the exact and symmetry adapted states using the critical points associated to the CS while at the right using the critical points of SAS.

Quantum phase transitions



Determination of the value of γ by means of the fidelity for $N = 20$ atoms and $\omega_A = 1$, we use a step of $\Delta\gamma = 0.001$.

The mimimum value of the fidelity is given by

$(\gamma, F(\gamma + \Delta\gamma, \gamma)) = (0.567, 0.999807)$.

Linear and VN Entropies

A pure bipartite state is entangled if it can not be written as a tensorial product of the two parts. To measure it, one uses:
(a) The linear entropy $S_L = 1 - Tr(\rho_M^2)$, and it takes the form

$$\begin{aligned} S_L^{(\pm)} &= 1 - N_\pm^4 \Big(\left(1 \pm e^{-2|\alpha|^2}\right)^2 \left(1 + (1 - |\zeta|^2])^{2j}\right)^2 \\ &+ \left(1 \mp e^{-2|\alpha|^2}\right)^2 \left(1 + (1 - |\zeta|^2)^{2j}\right)^2 \Big) \end{aligned}$$

(b) The von Neumann entropy

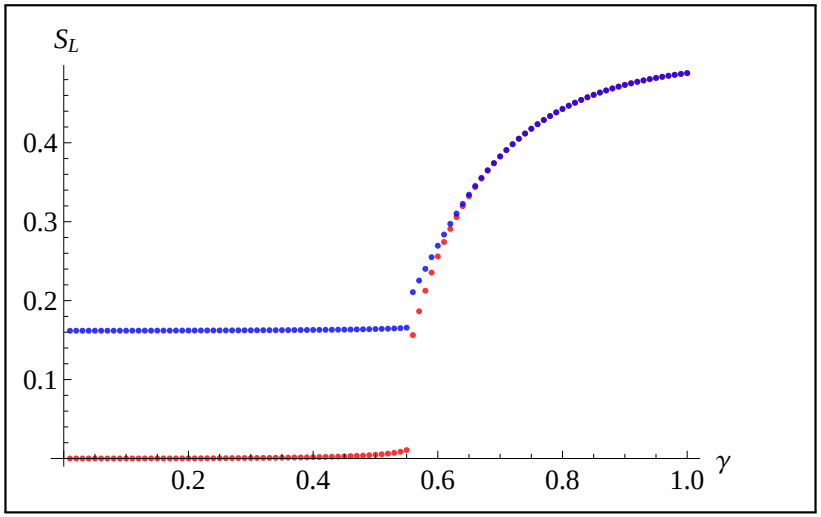
$$S_{VN}^{(\pm)} = -\lambda_+^{(\pm)} \log \lambda_+^{(\pm)} - \lambda_-^{(\pm)} \log \lambda_-^{(\pm)}$$

where

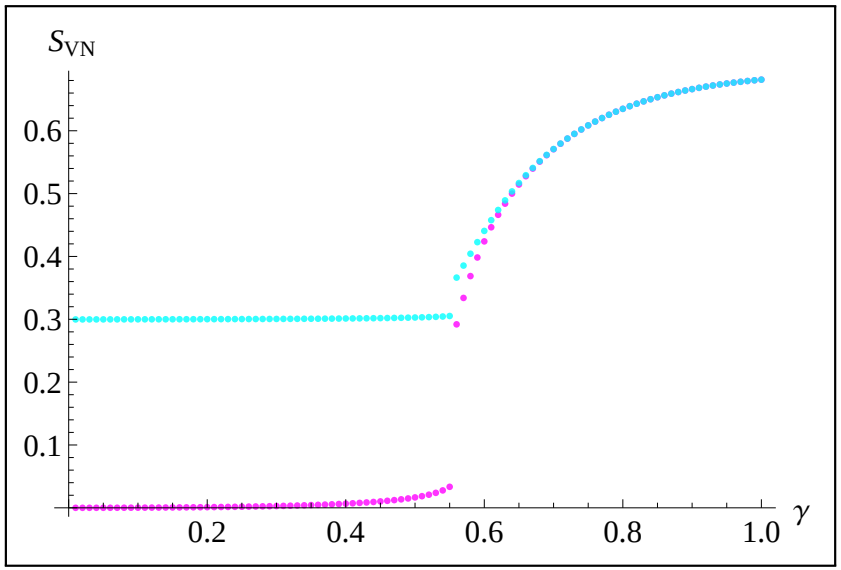
$$\lambda_+^{(\pm)} = N_\pm \left(1 \pm e^{-2|\alpha|^2}\right) \left(1 + (1 - |\zeta|^2])^{2j}\right)$$

$$\lambda_-^{(\pm)} = N_\pm \left(1 \mp e^{-2|\alpha|^2}\right) \left(1 - (1 - |\zeta|^2])^{2j}\right)$$

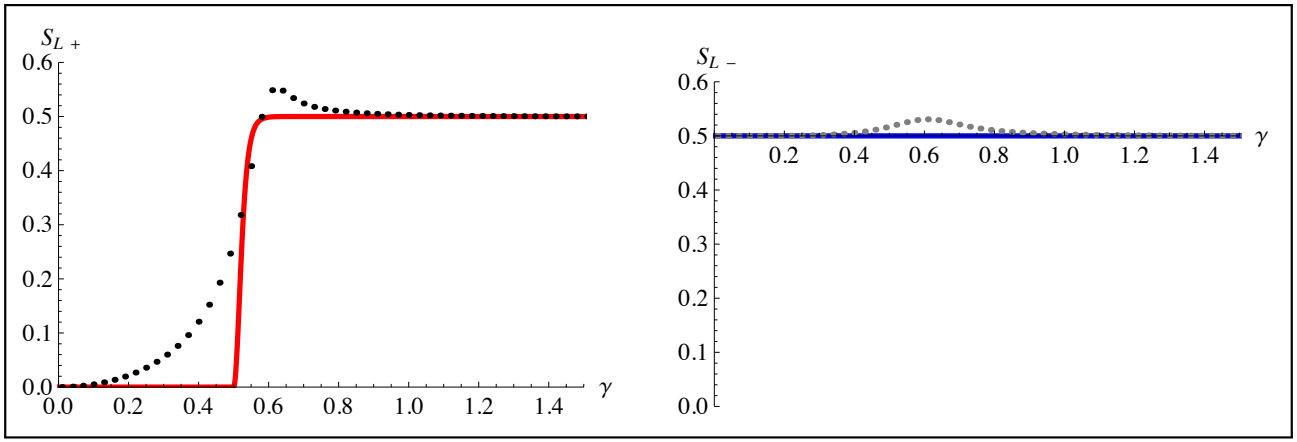
Linear and VN entropies for SAS



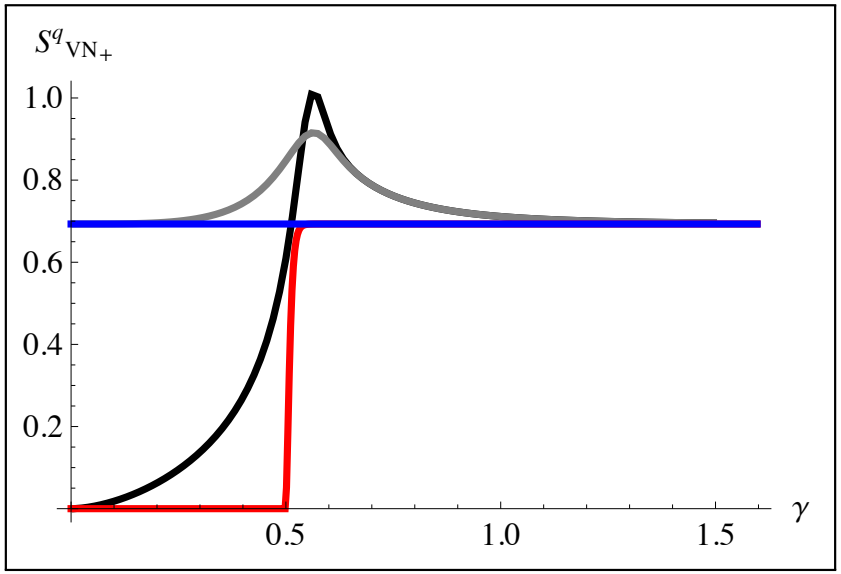
For $N = 20$ atoms and $\omega_A = 1$.



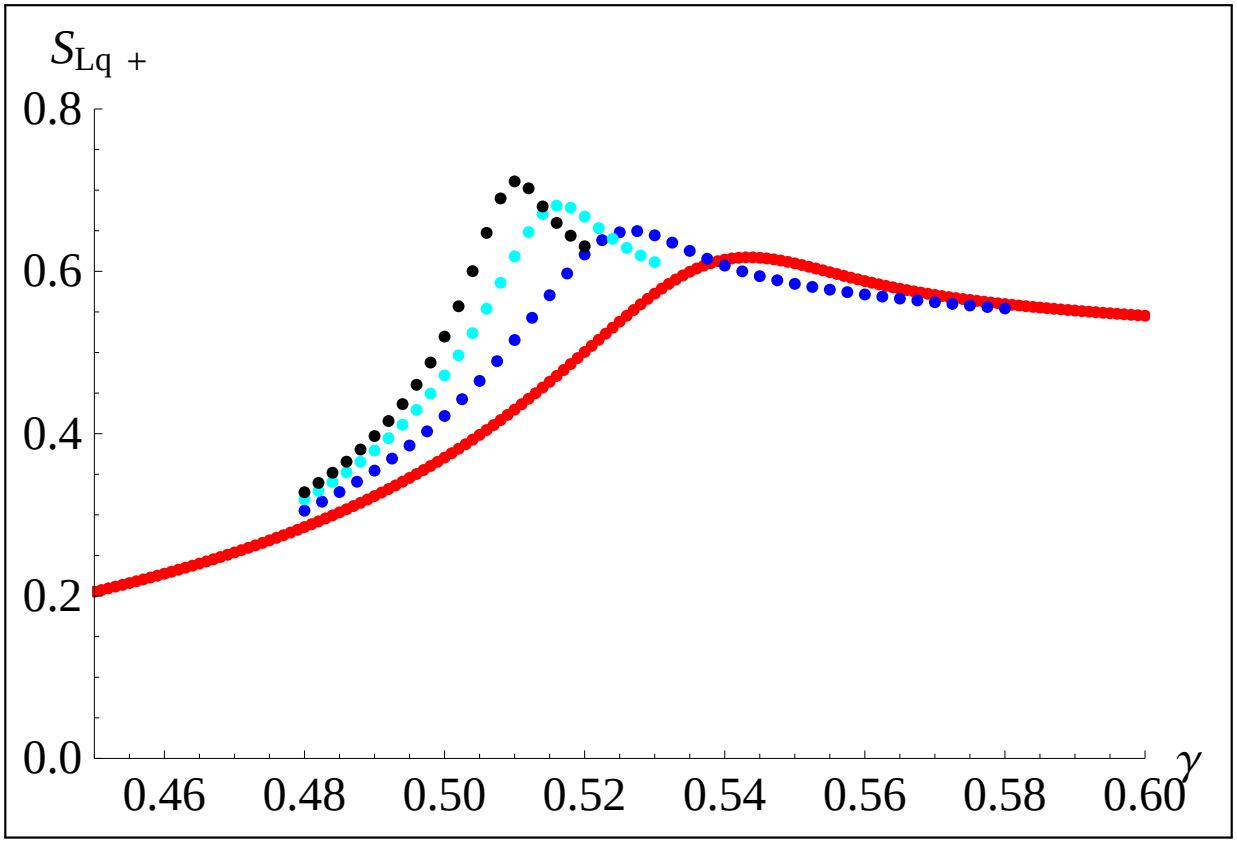
Comparison SAS vs Exact of linear and VN entropies



For $N = 10$ atoms and $\omega_A = 1$.



Linear Entropy for several number of atoms



We show the linear entropy as a function of γ for the following number of atoms: $N_A = 40, 80, 160,$ and 320.

Matter Squeezing Coefficients and Mandel parameter

The matter squeezing coefficients are defined as follows

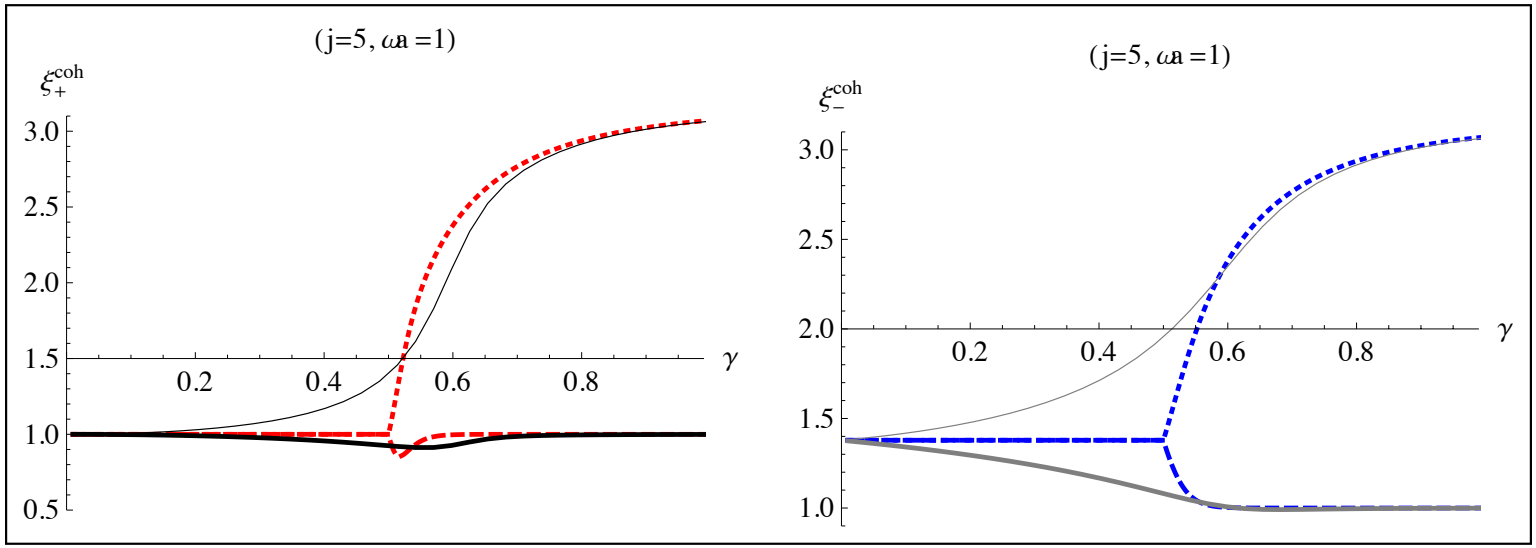
$$\xi = \sqrt{2}\,\frac{\Delta J_{\perp}}{\sqrt{|\langle \vec{J} \rangle|}}\,,$$

while for the photon part one has the parameter

$$Q = \frac{\Delta N_{ph}}{\langle N_{ph} \rangle} - 1\,.$$

For the CS one can prove that $\xi = 1$ and a Poisson statistics that is $Q = 0$

Matter and field statistics for SAS vs Exact



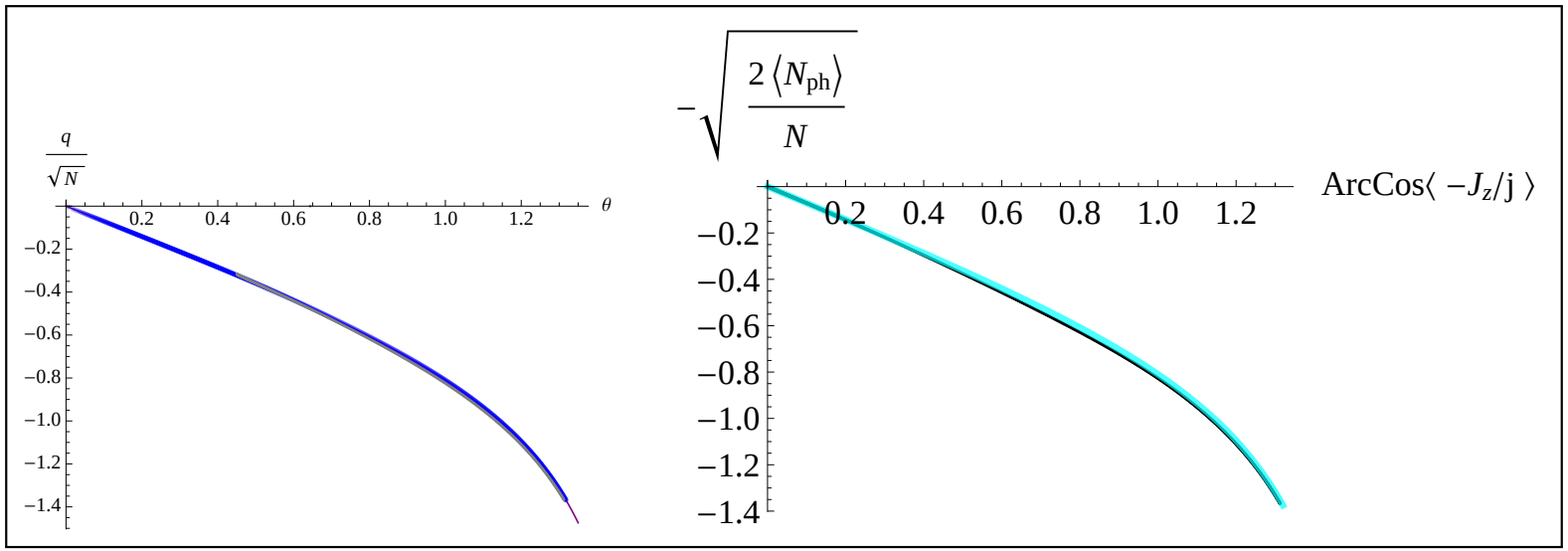
Matter squeezing parameters ξ_x (dashed) and ξ_y (dotted) for the even and odd SAS compared with the corresponding exact quantum results for $N = 10$ atoms and $\omega_A = 1$.

Universal invariant

The equivalent quantum operators to the phase space variables are q and θ are

$$q \longrightarrow \sqrt{2\langle \hat{a}^\dagger \hat{a}\rangle}\ ,$$

$$\theta \longrightarrow \arccos\left(-\frac{2}{N}\langle \hat{J}_z\rangle\right)\ .$$



All the plots fall in the same curve, the results are presented for $N = 10$ atoms and $\omega_A = 1$. However, it does not change for larger values of the number of atoms.

Conclusions

- The symmetry adapted states constitute an excellent approximation to the ground state and first excited state of the Dicke Model.
- We show that the CS approximation gives rise to significant differences with respect to the SAS case. The overlap or the fidelity of the variational SAS with respect to the exact solutions are close to one except in a small vicinity of the phase transition. For this reason, the fidelity is a very good tool to detect the position of the quantum phase transitions.
- We prove that most of the observables detect the quantum phase transition, the linear and von Neumann entropies also present singular values in that critical point. Besides we have shown that for a finite number of atoms the quantum phase transitions as a function of γ changes from its thermodynamical value.
- We prove that the curve

$$\sqrt{\frac{2}{N}\langle \hat{a}^{\dagger}\hat{a}\rangle} \text{ vs. } \arccos\left(-\frac{2}{N}\langle \hat{J}_z\rangle\right) ,$$

is a universal invariant for the Dicke model.

Thank you