Symmetries of neutrino interactions

A.B. Balantekin
University of Wisconsin-Madison

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Symmetries in Neutrino Physics

What we learned from Arima and Iachello

Symmetries characterize physics of many-body systems.

I will discuss how symmetry concept apply to neutrino physics using analogies to fermion-pairing.

Neutrino Mass

Dirac and Majorana masses

Dirac mass term:

$$H_m^D = m_D \int d^3x (\bar{\psi}_L \psi_R + h.c.)$$

Left- and right-handed Majorana mass terms:

$$\begin{aligned} H_m^M &= H_m^L + H_m^R \\ &= \frac{1}{2} m_L \int d^3x (\bar{\psi}_L \psi_L^c + h.c) + \frac{1}{2} m_R \int d^3x (\bar{\psi}_R \psi_R^c + h.c) \end{aligned}$$

$$\psi^C = C \bar{\psi}^T$$

Neutrino Mass

Dirac SU(2) algebra[$SU(2)_D$]

$$D_+ = \int d^3x(\bar{\psi}_L\psi_R)$$

$$D_- = \int d^3x(\bar{\psi}_R\psi_L) = D_+^\dagger$$

$$[D_+, D_-] = 2D_0$$

$$D_0 = \frac{1}{2}\int d^3x(\psi_L^\dagger\psi_L - \psi_R^\dagger\psi_R)$$

$$\boxed{[D_+, D_-] = 2D_0 \;\;,\;\; [D_0, D_+] = D_+ \;\;,\;\; [D_0, D_-] = -D_-}$$

Neutrino Mass

Majorana $SU(2)_L \times SU(2)_R$ algebra

$$L_+ = \frac{1}{2}\int d^3x(\bar{\psi}_L \psi_L^c) = L_-^\dagger, L_0 = \frac{1}{4}\int d^3x(\psi_L^\dagger \psi_L - \psi_L \psi_L^\dagger)$$

$$R_+ = \frac{1}{2}\int d^3x(\bar{\psi}_R^c \psi_R) = R_-^\dagger, R_0 = \frac{1}{4}\int d^3x(\psi_R \psi_R^\dagger - \psi_R^\dagger \psi_R)$$

$$\boxed{[L_+, L_-] = 2L_0 \;\;,\;\; [L_0, L_+] = L_+ \;\;,\;\; [L_0, L_-] = -L_-}$$

$$\boxed{[R_+, R_-] = 2R_0 \;\;,\;\; [R_0, R_+] = R_+ \;\;,\;\; [R_0, R_-] = -R_-}$$

$$[L_{-,+,0}, R_{-,+,0}] = 0, \quad D_0 \equiv L_0 + R_0$$

Neutrino Mass

Pauli-Gürsey SU(2), spanned by $A_{\pm}, A_0$:

$$
\begin{aligned}
[D_+, L_0] &= -\frac{1}{2} D_+ \quad , \quad [D_+, R_0] = -\frac{1}{2} D_+, \\
[D_+, L_+] &= 0 \quad , \quad [D_+, R_+] = 0, \\
[D_+, L_-] &= A_+ \quad , \quad [D_+, R_-] = A_-, \\
[D_-, L_0] &= \frac{1}{2} D_- \quad , \quad [D_-, R_0] = \frac{1}{2} D_-, \\
[D_-, L_+] &= -A_- \quad , \quad [D_-, R_+] = -A_+, \\
[D_-, L_-] &= 0 \quad , \quad [D_-, R_-] = 0
\end{aligned}
$$

$$
A_+ = \int d^3x \left[-\psi_L^T C \gamma_0 \psi_R \right], A_- = \int d^3x \left[\psi_R^\dagger \gamma_0 C (\psi_L^\dagger)^T \right].
$$

$$
\boxed{[A_+, A_-] = 2(R_0 - L_0) \equiv 2A_0, \; [A_+, A_0] = -A_+, \; [A_-, A_0] = A_-}
$$

Pauli-Gürsey symmetry

Particle-antiparticle symmetry realized via the Pauli-Gürsey transformation:

$$\psi \to \psi' = a\psi + b\gamma_5\psi^c$$

$$|a|^2 + |b|^2 = 1$$

The pairing algebra

SO(5)

The ten operators D_+, D_-, L_+, L_-, L_0, R_+, R_-, R_0, A_+, and A_- form a Lie algebra, the pairing algebra $SO(5)$.
The most general neutrino mass Hamiltonian:

$$H_m = m_D(D_+ + D_-) + m_L(L_+ + L_-) + m_R(R_+ + R_-).$$

sits in the $SO(5)/SU(2)_{\mathrm{PG}} \times U(1)_\chi$ coset where $U(1)_\chi$ is generated by $D_0 = L_0 + R_0$.

Sea-saw mechanism

$SU(2)_{\rm PG}$

The Pauli-Gürsey transformation: $\psi \rightarrow \psi' = a\psi + b\gamma_5\psi^c$.
Consider the most general element of the $SU(2)_{\rm PG}$ group:

$$\hat{U} = e^{-\tau^* A_-} e^{-log(1+|\tau|^2)A_0} e^{\tau A_+} e^{i\varphi A_0}$$

under which

$$\psi \rightarrow \psi' = \hat{U}\psi\hat{U}^\dagger = \frac{e^{i\varphi/2}}{\sqrt{1+|\tau|^2}}[\psi - \tau^*\gamma_5\psi^c]$$

This is a Pauli-Gürsey transformation with

$$a = \frac{e^{i\varphi/2}}{\sqrt{1+|\tau|^2}} \quad , \quad b = \frac{-\tau^* e^{i\varphi/2}}{\sqrt{1+|\tau|^2}}$$

Neutrino Mixing

Electroweak eigenstates are a combination of mass eigenstates:

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

When an electron neutrino is emitted from a nucleus (t=0), it is in the state

$$|\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle,$$

But, at time t, it evolves into

$$|\nu_e(t)\rangle = \cos\theta\, \mathrm{e}^{iE_1 t}|\nu_1\rangle + \sin\theta\, \mathrm{e}^{iE_2 t}|\nu_2\rangle$$

$$E_1 = \mathbf{p}^2 + m_1^2 \quad \text{and} \quad E_2 = \mathbf{p}^2 + m_2^2$$

Vacuum Oscillations

$$i\frac{\partial}{\partial t}\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \frac{\delta m^2}{4E}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix}$$

$$P(\nu_e \to \nu_e) = |\langle \nu_e|\nu_e\rangle|^2 = 1 - \sin^2 2\theta \sin^2\left(\frac{\delta m^2 L}{4E}\right)$$

Matter Effects

In vacuum: $E^2 = \mathbf{p}^2 + m^2$

In matter: $(E - V)^2 = \mathbf{p}^2 + m^2 \Rightarrow E^2 = \mathbf{p}^2 + m^2_{\rm eff}$

$$\text{with } m^2_{\rm eff} = m^2 + 2EV + \mathcal{O}(V^2)$$

$$i\frac{\partial}{\partial t}\begin{pmatrix} |\nu_e\rangle \\ |\nu\mu\rangle \end{pmatrix} = \begin{pmatrix} \varphi & \frac{\delta m^2}{4E}\sin 2\theta \\ \frac{\delta m^2}{4E}\sin 2\theta & -\varphi \end{pmatrix}\begin{pmatrix} |\nu_e\rangle \\ |\nu\mu\rangle \end{pmatrix}$$

$$\varphi = -\frac{\delta m^2}{4E}\cos 2\theta + \frac{1}{\sqrt{2}}G_F N_e$$

Neutrino Mixing with 3 flavors

$$\mathbf{T} = \mathbf{T}_{23}\mathbf{T}_{13}\mathbf{T}_{12}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{23} & S_{23} \\ 0 & -S_{23} & C_{23} \end{pmatrix}}_{\mathbf{T}_{23}} \overbrace{\begin{pmatrix} C_{13} & 0 & S_{13}e^{-i\delta_{CP}} \\ 0 & 1 & 0 \\ -S_{13}e^{i\delta_{CP}} & 0 & C_{13} \end{pmatrix}}^{\mathbf{T}_{13}} \underbrace{\begin{pmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{T}_{12}}$$

$C_{ij} = \cos\theta_{ij}$, $S_{ij} = \sin\theta_{ij}$
δ_{CP}: CP-violating phase

Neutrino Flavor Evolution

MSW Equations

$$i\frac{\partial}{\partial t}\begin{pmatrix}\Psi_e\\ \Psi_\mu\\ \Psi_\tau\end{pmatrix} = \mathbf{H}\begin{pmatrix}\Psi_e\\ \Psi_\mu\\ \Psi_\tau\end{pmatrix}$$

$$\mathbf{H} = \mathbf{T}_{23}\mathbf{T}_{13}\mathbf{T}_{12}\begin{pmatrix}E_1 & 0 & 0\\ 0 & E_2 & 0\\ 0 & 0 & E_3\end{pmatrix}\mathbf{T}_{12}^\dagger\mathbf{T}_{13}^\dagger\mathbf{T}_{23}^\dagger + \begin{pmatrix}V_e & 0 & 0\\ 0 & V_\mu & 0\\ 0 & 0 & V_\tau\end{pmatrix}$$

Neutrino Flavor Evolution

MSW Equations

$$\tilde{\Psi}_\mu = \cos\theta_{23}\Psi_\mu - \sin\theta_{23}\Psi_\tau,$$

$$\tilde{\Psi}_\tau = \sin\theta_{23}\Psi_\mu + \cos\theta_{23}\Psi_\tau,$$

$$i\frac{\partial}{\partial t}\begin{pmatrix}\Psi_e\\ \tilde{\Psi}_\mu\\ \tilde{\Psi}_\tau\end{pmatrix} = \tilde{\mathbf{H}}\begin{pmatrix}\Psi_e\\ \tilde{\Psi}_\mu\\ \tilde{\Psi}_\tau\end{pmatrix}$$

$$\tilde{\mathbf{H}} = \mathbf{T}_{13}\mathbf{T}_{12}\begin{pmatrix}E_1 & 0 & 0\\ 0 & E_2 & 0\\ 0 & 0 & E_3\end{pmatrix}\mathbf{T}_{12}^\dagger\mathbf{T}_{13}^\dagger + \mathbf{T}_{23}^\dagger\begin{pmatrix}V_e & 0 & 0\\ 0 & V_\mu & 0\\ 0 & 0 & V_\tau\end{pmatrix}\mathbf{T}_{23}$$

Neutrino Flavor Evolution

At the *tree level*, the neutral current contribution to all the potentials are the same, but there are *differences* coming from the loop diagrams! There is an additional charged-current contribution to V_e. Taking out an overall phase, one only needs $V_{e\mu} = V_e - V_\mu$ and $V_{\tau\mu} = V_\tau - V_\mu$:

$$\tilde{\mathbf{H}} = \mathbf{T}_r \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} \mathbf{T}_r^\dagger + \begin{pmatrix} V_{e\mu} & 0 & 0 \\ 0 & S_{23}^2 V_{\tau\mu} & -C_{23} S_{23} V_{\tau\mu} \\ 0 & -C_{23} S_{23} V_{\tau\mu} & C_{23}^2 V_{\tau\mu} \end{pmatrix}$$

$$\mathbf{T}_r = \mathbf{T}_{13} \mathbf{T}_{12}$$

Neutrino Flavor Evolution

One-Body Hamiltonian

$$\tilde{\mathbf{H}} = \mathbf{T}_r \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} \mathbf{T}_r^\dagger + \begin{pmatrix} V_{e\mu} & 0 & 0 \\ 0 & S_{23}^2 V_{\tau\mu} & -C_{23} S_{23} V_{\tau\mu} \\ 0 & -C_{23} S_{23} V_{\tau\mu} & C_{23}^2 V_{\tau\mu} \end{pmatrix}$$

Dominant term, Wolfenstein

$$V_{e\mu}(x) = \sqrt{2} G_F N_e(x)$$

Sub-dominant term, Botella, Lim, Marciano, PRD **35**, 896 (1987)

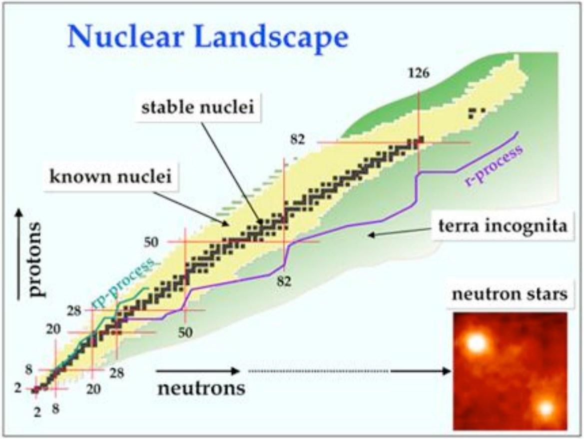
$$V_{\tau\mu} = -\frac{3\sqrt{2} G_F \alpha}{\pi \sin^2\theta_W} \left(\frac{m_\tau}{m_W}\right)^2 \left\{ (N_e + N_n) \log \frac{m_\tau}{m_W} + \left(\frac{N_e}{2} + \frac{N_n}{3}\right) \right\}$$

Motivation

Supernova neutrinos

- $M_{\rm progenitor} \geq 8 M_{\odot} \Rightarrow \Delta E \sim 10^{59}$ MeV
- 99 % of this energy is carried away by neutrinos and antineutrinos with $10 \leq E_{\nu} \leq 30$ MeV $\Rightarrow 10^{58}$ neutrinos!

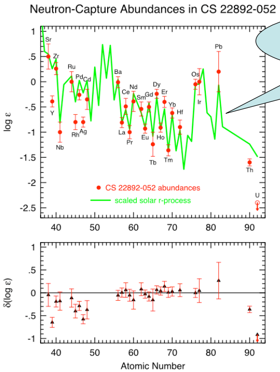
r-process Nucleosynthesis



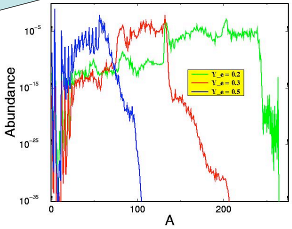
r-process Nucleosynthesis

[Fe/H] ≈ -3.1

r-process abundances



A > 100 abundance pattern fits the solar abundances well



r-process abundances should depend very strongly on electron fraction Meyer

r-process Nucleosynthesis

- Yields of r-process nucleosynthesis are determined by the electron fraction, or equivalently by the neutron-to-proton ratio, n/p
- Interactions of the neutrinos and antineutrinos streaming out of the core both with nucleons and seed nuclei determine the n/p ratio. Hence it is crucial to understand neutrino properties and interactions.
- As these neutrinos reach the r-process region they undergo matter-enhanced neutrino oscillations as well as coherently scatter over other neutrinos. Many-body behavior of this neutrino gas is still being explored, but may have significant impact on r-process nucleosynthesis.

Neutrino Mixing

Mass and Flavor States

$$a_1(\mathbf{p}, s) = \cos\theta \; a_e(\mathbf{p}, s) - \sin\theta \; a_x(\mathbf{p}, s)$$

$$a_2(\mathbf{p}, s) = \sin\theta \; a_e(\mathbf{p}, s) + \cos\theta \; a_x(\mathbf{p}, s)$$

Flavor Isospin Operators

$$\hat{J}^+_{\mathbf{p},s} = a_e^\dagger(\mathbf{p}, s) a_x(\mathbf{p}, s) \;, \qquad \hat{J}^-_{\mathbf{p},s} = a_x^\dagger(\mathbf{p}, s) a_e(\mathbf{p}, s) \;,$$

$$\hat{J}^0_{\mathbf{p},s} = \frac{1}{2}\left(a_e^\dagger(\mathbf{p}, s) a_e(\mathbf{p}, s) - a_x^\dagger(\mathbf{p}, s) a_x(\mathbf{p}, s) \right)$$

$$[\hat{J}^+_{\mathbf{p},s}, \hat{J}^-_{\mathbf{q},r}] = 2\delta_{\mathbf{pq}}\delta_{sr}\hat{J}^0_{\mathbf{p},s} \;, \qquad [\hat{J}^0_{\mathbf{p},s}, \hat{J}^\pm_{\mathbf{q},r}] = \pm\delta_{\mathbf{pq}}\delta_{sr}\hat{J}^\pm_{\mathbf{p},s} \;,$$

Neutrino Hamiltonian

Vacuum Oscillation Term

$$\hat{H}_\nu^{(1)} = \sum_{\mathbf{p},s} \left(\frac{m_1^2}{2p} a_1^\dagger(\mathbf{p},s) a_1(\mathbf{p},s) + \frac{m_2^2}{2p} a_2^\dagger(\mathbf{p},s) a_2(\mathbf{p},s) \right) .$$

$$\hat{H}_\nu^{(1)} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p$$

$$\hat{B} = (\sin 2\theta, 0, -\cos 2\theta)$$

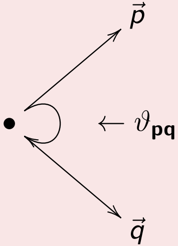
One-Body Hamiltonian including interactions with the electron background

$$\hat{H}_\nu = \sum_p \left(\frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p - \sqrt{2} G_F N_e J_p^0 \right)$$

Neutrino Hamiltonian

Neutrino-Neutrino Interactions

$$\hat{H}_{\nu\nu} = \frac{\sqrt{2}G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos\vartheta_{\mathbf{pq}})\, \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$



$(1 - cos\vartheta)$ terms follow from the V-A nature of the weak interactions.

Neutrino Hamiltonian

The total neutrino Hamiltonian

$$\hat{H}_{\text{total}} = H_\nu + H_{\nu\nu} = \left(\sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p - \sqrt{2} G_F N_e J_p^0 \right) + \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos\vartheta_{\mathbf{pq}}) \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

Pantaleone, Dasgupta, Duan, Fogli, Fuller, Kostelecky, McKellar, Lisi, Mirizzi, Qian, Pastor, Raffelt, Samuel, Sawyer, Sigl, Smirnov, . . .

Evolution Equations

Path Integral for the Evolution Operator

$$i\frac{\partial U}{\partial t} = \left(H_\nu + H_{\nu\nu}\right) U$$

Use SU(2) coherent states to write the evolution operator as a path integral:

$$|z(t)\rangle = \exp\left(\int dp z(p,t) J_+(p)\right)|\phi\rangle$$

$$|\phi\rangle = \prod_p a_e^\dagger(p)|0\rangle$$

$$\langle z'(t_f)|U|z(t_i)\rangle = \int \mathcal{D}[z,z^*]\exp\left(iS[z,z^*]\right)$$

Calculating the Evolution Operator

Stationary Phase Approximation

$$\langle z'(t_f)|U|z(t_i)\rangle = \int \mathcal{D}[z,z^*] \exp\left(iS[z,z^*]\right)$$

$$S(z,z^*) = \int_{t_i}^{t_f} dt \frac{\langle z(t)|i\frac{\partial}{\partial t} - H(t)|z(t)\rangle}{\langle z(t)|z(t)\rangle} + \log\langle z'(t_f)|z(t_f)\rangle$$

$$H = H_\nu + H_{\nu\nu}$$

$$\left(\frac{d}{dt}\frac{\partial}{\partial \dot{z}} - \frac{\partial}{\partial z}\right) L(z,z^*) = 0 \qquad \left(\frac{d}{dt}\frac{\partial}{\partial \dot{z}^*} - \frac{\partial}{\partial z^*}\right) L(z,z^*) = 0$$

$$i\dot{z}(p,t) = \beta(p,t) - \alpha(p,t)z(p,t) - \beta^*(p,t)z(p,t)^2$$

$$\alpha(p,t) = -\frac{\delta m^2}{2p}\cos 2\theta + \sqrt{2}G_F N_e + \sqrt{2}G_F \int dq(1-\cos\theta_{pq})\left(\frac{1-|z(q,t)|^2}{1+|z(q,t)|^2}\right)$$

$$\beta(p,t) = \frac{1}{2}\frac{\delta m^2}{2p}\sin 2\theta + \sqrt{2}G_F \int dq(1-\cos\theta_{pq})\left(\frac{z(q,t)}{1+|z(q,t)|^2}\right)$$

$$z(p,t) = \frac{\psi_x(p,t)}{\psi_e(p,t)}, \quad |\psi_e|^2 + |\psi_x|^2 = 1$$

The mean-field/RPA solution

$$z(p,t)=\frac{\psi_x(p,t)}{\psi_e(p,t)},\quad |\psi_e|^2+|\psi_x|^2=1$$

$$\Delta=\frac{\delta m^2}{2p},\qquad A=\sqrt{2}G_F N_e$$

$$D=\sqrt{2}G_F\int dq(1-\cos\theta_{pq})\left[\left(|\psi_e(q,t)|^2-|\psi_x(q,t)|^2\right)\right]$$

$$D_{ex}=2\sqrt{2}G_F\int dq(1-\cos\theta_{pq})\left(\psi_e(q,t)\psi_x^*(q,t)\right)$$

$$i\frac{\partial}{\partial t}\begin{pmatrix}\psi_e\\ \psi_x\end{pmatrix}=\frac{1}{2}\begin{pmatrix}A+D-\Delta\cos 2\theta & D_{e\mu}+\Delta\sin 2\theta\\ D_{\mu e}+\Delta\sin 2\theta & -A-D+\Delta\cos 2\theta\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_x\end{pmatrix}$$

This is the mean field approximation! Recall that one can approximate product of two commuting arbitrary operators $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$ as

$$\hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \sim \hat{\mathcal{O}}_1 \langle \xi | \hat{\mathcal{O}}_2 | \xi \rangle + \langle \xi | \hat{\mathcal{O}}_1 | \xi \rangle \hat{\mathcal{O}}_2 - \langle \xi | \hat{\mathcal{O}}_1 | \xi \rangle \langle \xi | \hat{\mathcal{O}}_2 | \xi \rangle,$$

provided that

$$\langle \xi | \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 | \xi \rangle = \langle \xi | \hat{\mathcal{O}}_1 | \xi \rangle \langle \xi | \hat{\mathcal{O}}_2 | \xi \rangle.$$

This reduces $H_{\nu\nu}$ to a one-body Hamiltonian:

$$\begin{aligned} \mathcal{H}_{\nu\nu} &\sim 2\frac{\sqrt{2}G_F}{V} \int d^3p \, d^3q \, R_{pq} \, \Big(J_0(p) \langle J_0(q) \rangle \\ &+ \frac{1}{2} J_+(p) \langle J_-(q) \rangle + \frac{1}{2} J_-(p) \langle J_+(q) \rangle \Big) \end{aligned}$$

Beyond the Mean field

Corrections to RPA

$$\beta(p,t) = \frac{z(p,t)}{\sqrt{1+|z(p,t)|^2}} \qquad \beta^*(p,t) = \frac{z^*(p,t)}{\sqrt{1+|z(p,t)|^2}}$$

$$\langle z'(t_f)|U|z(t_i)\rangle = \int \lim_{N\to\infty} \prod_{\alpha=1}^{N} \prod_{p\in\mathcal{P}} \frac{d\beta(p,t_\alpha)d\beta^*(p,t_\alpha)}{i\pi} e^{iS[\beta,\beta^*]}$$

$$\langle z'(t_f)|U|z(t_i)\rangle = \lim_{N\to\infty} (i\pi)^{N+P} \frac{e^{iS[\beta_{cl},\beta^*_{cl}]}}{\sqrt{Det\,(KM - L^T K^{-1} L)}}$$

$P =$ the number of allowed momentum modes.

$$K(p,k,q,m) = \frac{1}{2}\left(\frac{\delta^2 S}{\delta x(p,t_k)\,\delta x(q,t_m)}\right)_{cl}$$

$$M(p,k,q,m) = \frac{1}{2}\left(\frac{\delta^2 S}{\delta y(p,t_k)\,\delta y(q,t_m)}\right)_{cl}$$

$$L(p,k,q,m) = \frac{1}{2}\left(\frac{\delta^2 S}{\delta x(p,t_k)\,\delta y(q,t_m)}\right)_{cl}$$

$x = (\tilde\beta + \tilde\beta^*)/2,\ y = (\tilde\beta - \tilde\beta^*)/2i$
$\tilde\beta = \beta - \beta_{cl}$

Antineutrinos and three flavors

Including antineutrinos

$$H = H_\nu + H_{\bar{\nu}} + H_{\nu\nu} + H_{\bar{\nu}\bar{\nu}} + H_{\nu\bar{\nu}}$$

Requires introduction of a second set of SU(2) algebras!

Including three flavors

Requires introduction of SU(3) algebras.

Both extensions are straightforward, but tedious!

Polarization vector

Polarization vector

$$P_i(q) = \mathrm{Tr}(J_i(q)\rho)$$

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{ex} \\ \rho_{xe} & \rho_{xx} \end{pmatrix} = \frac{1}{2}\left(P_0 + \mathbf{P}\cdot\sigma\right)$$

$$\partial_r \mathbf{P}_p = \left\{ +\Delta + \sqrt{2}\,G_{\mathrm{F}}\left[N_e \hat{\mathbf{z}} + \int d\mathbf{q}\,(1-\cos\theta_{pq})\,(\mathbf{P}_{\mathbf{q}} - \overline{\mathbf{P}}_{\mathbf{q}}) \right] \right\} \times \mathbf{P}_p$$

$$\partial_r \overline{\mathbf{P}}_p = \left\{ -\Delta + \sqrt{2}\,G_{\mathrm{F}}\left[N_e \hat{\mathbf{z}} + \int d\mathbf{q}\,(1-\cos\theta_{pq})\,(\mathbf{P}_{\mathbf{q}} - \overline{\mathbf{P}}_{\mathbf{q}}) \right] \right\} \times \overline{\mathbf{P}}_p$$

$$\Delta = \frac{\delta m^2}{2p}(\sin 2\theta \hat{\mathbf{x}} - \cos 2\theta \hat{\mathbf{z}})$$

Neutrino Hamiltonian

Neutrino Hamiltonian with $\nu - \nu$ interactions

$$\hat{H}_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos\vartheta_{\mathbf{pq}}) \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

Single-angle approximation $\Rightarrow$

$$\hat{H}_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \vec{J} \cdot \vec{J}$$

Defining $\mu = \frac{\sqrt{2} G_F}{V}$, $\tau = \mu t$, and $\omega_p = \frac{1}{\mu} \frac{\delta m^2}{2p}$ one can write

$$\hat{H} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{J} \cdot \vec{J}$$

Conserved Quantities

Some Invariants

$$\hat{H} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{J} \cdot \vec{J}$$

This Hamiltonian preserves the *length of each spin*

$$\hat{L}_p = \vec{J}_p \cdot \vec{J}_p \; , \qquad \left[\hat{H}, \hat{L}_p \right] = 0 \; ,$$

as well as the *total spin component* in the direction of the "external magnetic field", $\hat{B}$

$$\hat{C}_0 = \hat{B} \cdot \vec{J} \; , \qquad \left[\hat{H}, \hat{C}_0 \right] = 0$$

Raffelt, Smirnov, Fuller, Pehlivan, Balantekin, Kajino, Yoshida, $\cdots$

BCS Hamiltonian

Hamiltonian in Quasi-spin basis

$$\hat{H}_{\rm BCS} = \sum_k 2\epsilon_k \hat{t}^0_k - |G| \hat{T}^+ \hat{T}^.$$

Quasi-spin operators:

$$\hat{t}^+_k = c^\dagger_{k\uparrow} c^\dagger_{k\downarrow}, \qquad \hat{t}^-_k = c_{k\downarrow} c_{k\uparrow}, \qquad \hat{t}^0_k = \frac{1}{2}\left(c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{k\downarrow} c_{k\downarrow} - 1 \right)$$

$$[\hat{t}^+_k, \hat{t}^-_l] = 2\delta_{kl}\hat{t}^0_k \ , \qquad [\hat{t}^0_k, \hat{t}^\pm_l] = \pm\delta_{kl}\hat{t}^\pm_k \ .$$

Richardson gave a solution of this problem. Hence there exist invariants of motion.

The duality between $\nu - \nu$ and BCS Hamiltonians

The ν-ν Hamiltonian

$$\hat{H} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{J} \cdot \vec{J}$$

$\Longleftrightarrow$

The BCS Hamiltonian

$$\hat{H}_{\mathrm{BCS}} = \sum_k 2\epsilon_k \hat{t}_k^0 - |G| \hat{T}^+ \hat{T}$$

Same symmetries leading to Analogous (dual) dynamics!

Invariants

Invariants

The collective neutrino Hamiltonian given has the following constants of motion:

$$\hat{h}_p = \hat{B} \cdot \vec{J}_p + 2 \sum_{q(\neq p)} \frac{\vec{J}_p \cdot \vec{J}_q}{\omega_p - \omega_q}.$$

The individual neutrino spin-length discussed before in an independent invariant. However $\hat{C}_0 = \sum_p \hat{h}_p$. The Hamiltonian itself is also a linear combination of these invariants.

$$\hat{H} = \sum_p w_p \hat{h}_p + \sum_p \hat{L}_p \ .$$

Pehlivan, Balantekin, Kajino, Yoshida

Eigenvalues and Eigenstates

Eigenstates of the system

- $J_{\max} = N/2$ $\qquad$ N, the total number of neutrinos
- A state with all electron neutrinos:
  $|\nu_e \, \nu_e \, \nu_e \, \ldots \rangle = |J_{\max} \;\; J_{\max}\rangle_f$
- Matter and flavor bases are connected with a unitary transformation: $|J_{\max} \;\; J_{\max}\rangle_f = \hat{U}^\dagger |J_{\max} \;\; J_{\max}\rangle_m$
- $|J_{\max} \;\; J_{\max}\rangle_m = \prod_{\mathbf{p},s} a_1^\dagger(\mathbf{p},s) \, |0\rangle$
  $|J_{\max} \;\; -J_{\max}\rangle_m = \prod_{\mathbf{p},s} a_2^\dagger(\mathbf{p},s) \, |0\rangle$
  $E_{(+J_{\max})} = -\sum_p \frac{n_p}{2} \omega_p + J_{\max}\,(J_{\max}+1)$
  $E_{(-J_{\max})} = \sum_p \frac{n_p}{2} \omega_p + J_{\max}\,(J_{\max}+1)$

Eigenvalues and Eigenstates

Other states

$$\mathcal{Q}^{\pm}(\xi) == \sum_p \frac{1}{\omega_p - \xi}\left(\cos^2\theta \hat{J}_p^{\pm} + \sin 2\theta \hat{J}_p^0 - \sin^2\theta \hat{J}_p^{\mp}\right)$$

$$\begin{aligned} \hat{H}\mathcal{Q}^{+}(\xi)|J \; -J\rangle_m &= \left(E_{(-J)} - 2J - \xi\right)\mathcal{Q}^{+}(\xi)|J \; -J\rangle_m \\ &+ \underbrace{\left(1 + 2\sum_p \frac{-j_p}{w_p - \xi}\right)\mathcal{Q}^{+}|J \; -J\rangle_m}_{\text{should be zero if eigenstate}} \end{aligned}$$

This gives us the Bethe ansatz equation $\Rightarrow \sum_p \frac{-j_p}{w_p - \xi} = -\frac{1}{2}$

Eigenvalues and Eigenstates

Most General Eigenstate

$$|\xi_1, \xi_2, \dots \xi_\kappa\rangle \equiv \mathcal{Q}^+(\xi_1)\mathcal{Q}^+(\xi_2)\dots\mathcal{Q}^+(\xi_\kappa)|J \ -J\rangle_m$$

$$E(\xi_1, \xi_2, \dots, \xi_\kappa) = E_{(-J)} - \sum_{\alpha=1}^{\kappa} \xi_\alpha - \kappa(2J - \kappa + 1) \ ,$$

$$\underbrace{\sum_p \frac{-j_p}{\omega_p - \xi_\alpha} = -\frac{1}{2} + \sum_{\substack{\beta=1 \\ (\beta \neq \alpha)}}^{\kappa} \frac{1}{\xi_\alpha - \xi_\beta} \ .}_{\text{Bethe ansatz equations}}$$

An RPA-like approximation

An RPA-inspired approximation when $[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2] = 0$. Approximate the operator product as

$$\hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \sim \hat{\mathcal{O}}_1 \langle \hat{\mathcal{O}}_2 \rangle + \langle \hat{\mathcal{O}}_1 \rangle \hat{\mathcal{O}}_2 - \langle \hat{\mathcal{O}}_1 \rangle \langle \hat{\mathcal{O}}_2 \rangle \ ,$$

where the expectation values should be calculated with respect to a state $|\Psi\rangle$ which satisfies the condition $\langle \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \rangle = \langle \hat{\mathcal{O}}_1 \rangle \langle \hat{\mathcal{O}}_2 \rangle$.

$$\hat{H} \sim \hat{H}^{\mathrm{RPA}} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{P} \cdot \vec{J}$$

Polarization vector: $\vec{P}_{\mathbf{p},s} = 2\langle \vec{J}_{\mathbf{p},s} \rangle$. Use SU(2) coherent states for the expectation value.

Mean-neutrino field

Polarization vectors

$$\hat{H} \sim \hat{H}^{\mathrm{RPA}} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{P} \cdot \vec{J}$$

$$\vec{P}_{\mathbf{p},s} = 2\langle \vec{J}_{\mathbf{p},s} \rangle$$

Eqs. of motion: $$\frac{d}{d\tau}\vec{J}_p = -i[\vec{J}_p, \hat{H}^{\mathrm{RPA}}] = (\omega_p \hat{B} + \vec{P}) \times \vec{J}_p$$

RPA Consistency requirement $\Rightarrow \frac{d}{d\tau}\vec{P}_p = (\omega_p \hat{B} + \vec{P}) \times \vec{P}_p$

Invariants $$I_p = 2\langle \hat{h}_p \rangle = \hat{B} \cdot \vec{P}_p + \sum_{q(\neq p)} \frac{\vec{P}_p \cdot \vec{P}_q}{\omega_p - \omega_q} \Rightarrow \frac{d}{d\tau} I_p = 0$$

Raffelt; Pehlivan, Balantekin, Kajino, Yoshida

Total Hamiltonian

Hamiltonian with both ν's and $\bar{\nu}$'s

$$
\begin{aligned}
\hat{H}_{\text{total}} &= \sum_{p} \frac{\delta m^2}{2p} \left(-\cos 2\theta \, \hat{J}^0_p + \sin 2\theta \, \frac{\hat{J}^+_p + \hat{J}^-_p}{2} \right) \\
&+ \sum_{\bar{p}} \frac{\delta m^2}{2\bar{p}} \left(\cos 2\theta \, \hat{\bar{J}}^0_{\bar{p}} + \sin 2\theta \, \frac{\hat{\bar{J}}^+_{\bar{p}} + \hat{\bar{J}}^-_{\bar{p}}}{2} \right) \\
&+ \frac{\sqrt{2} G_F}{V} \left(\sum_{\mathbf{p},\mathbf{q}} (1 - \cos \vartheta_{\mathbf{pq}}) \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}} + \sum_{\bar{\mathbf{p}},\bar{\mathbf{q}}} (1 - \cos \vartheta_{\bar{\mathbf{p}}\bar{\mathbf{q}}}) \vec{\bar{J}}_{\bar{\mathbf{p}}} \cdot \vec{\bar{J}}_{\bar{\mathbf{q}}} \right. \\
&+ \left. \sum_{\mathbf{p},\bar{\mathbf{q}}} (1 - \cos \vartheta_{\mathbf{p}\bar{\mathbf{q}}}) \left(2 \hat{J}^0_{\mathbf{p}} \hat{\bar{J}}^0_{\bar{\mathbf{q}}} - \hat{J}^+_{\mathbf{p}} \hat{\bar{J}}^-_{\bar{\mathbf{q}}} - \hat{J}^-_{\mathbf{p}} \hat{\bar{J}}^+_{\bar{\mathbf{q}}} \right) \right) .
\end{aligned}
$$

Including antineutrinos

Single angle approximation

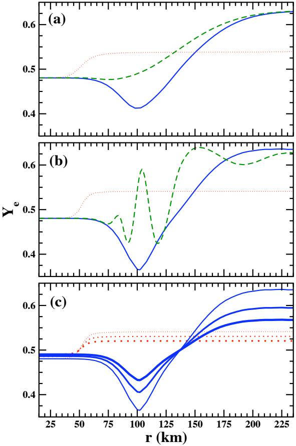
$$H_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p - \sum_{\bar{p}} \frac{\delta m^2}{2\bar{p}} \hat{B} \cdot \vec{\tilde{J}}_p + \frac{\sqrt{2} G_F}{V} \left(\vec{J} + \vec{\tilde{J}} \right) \cdot \left(\vec{J} + \vec{\tilde{J}} \right)$$

Defining $\omega_{\bar{p}} = -\frac{1}{\mu} \frac{\delta m^2}{2\bar{p}}$, one writes

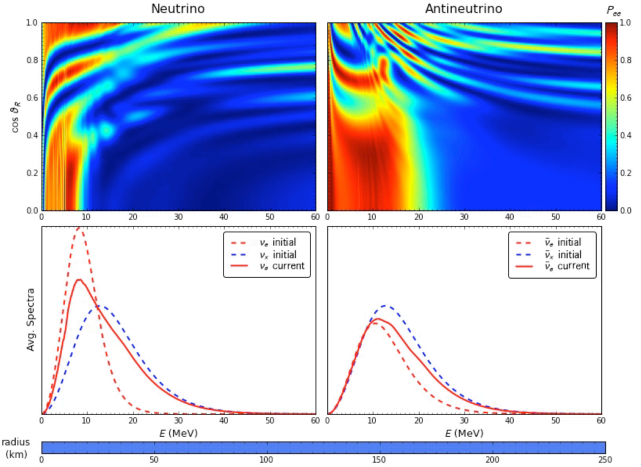
$$H = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \sum_{\bar{p}} \omega_{\bar{p}} \hat{B} \cdot \vec{\tilde{J}}_p + \left(\vec{J} + \vec{\tilde{J}} \right) \cdot \left(\vec{J} + \vec{\tilde{J}} \right)$$

Examples of mean-field calculations

- With ν luminosity $L^{51} =$ 0.001 (blue), 0.1 (green), 50 (red)
- Balantekin and Yüksel, New J. Phys. **7** 51 (2005).



Examples of mean-field calculations



Fuller, Qian, Carlson, Duan

Invariants

Invariants

Conserved quantities for each neutrino energy mode p:

$$\hat{h}_p = \hat{B} \cdot \vec{J}_p + 2 \sum_{q(\neq p)} \frac{\vec{J}_p \cdot \vec{J}_q}{\omega_p - \omega_q} + 2 \sum_{\bar{q}} \frac{\vec{J}_p \cdot \vec{\tilde{J}}_{\bar{q}}}{\omega_p - \omega_{\bar{q}}}$$

Conserved quantity $\hat{h}_{\bar{p}}$ for each antineutrino energy mode:

$$\hat{h}_{\bar{p}} = \hat{B} \cdot \vec{\tilde{J}}_p + 2 \sum_{\bar{q}(\neq \bar{p})} \frac{\vec{\tilde{J}}_{\bar{p}} \cdot \vec{\tilde{J}}_{\bar{q}}}{\omega_{\bar{p}} - \omega_{\bar{q}}} + 2 \sum_{q} \frac{\vec{\tilde{J}}_{\bar{p}} \cdot \vec{J}_q}{\omega_{\bar{p}} - \omega_q} \ .$$

Invariants

Mean-field Invariants

$$I_p = 2\langle \hat{h}_p \rangle = \hat{B} \cdot \vec{P}_p + \sum_{q(\neq p)} \frac{\vec{P}_p \cdot \vec{P}_q}{\omega_p - \omega_q} + \sum_{\bar{q}} \frac{\vec{P}_p \cdot \vec{\tilde{P}}_{\bar{q}}}{\omega_p - \omega_{\bar{q}}}$$

$$I_{\bar{p}} = 2\langle \hat{h}_{\bar{p}} \rangle = \hat{B} \cdot \vec{\tilde{P}}_{\bar{p}} + \sum_{\bar{q}(\neq \bar{p})} \frac{\vec{\tilde{P}}_{\bar{p}} \cdot \vec{\tilde{P}}_{\bar{q}}}{\omega_{\bar{p}} - \omega_{\bar{q}}} + \sum_{q} \frac{\vec{\tilde{P}}_{\bar{p}} \cdot \vec{P}_q}{\omega_{\bar{p}} - \omega_q}$$

Raffelt; Pehlivan *et al.*

Spectral Splits

Lagrange multiplier to enforce neutrino number conservation:

$$\hat{H}^{\mathrm{RPA}} + \omega_c \hat{J}^0 = \sum_p (\omega_c - \omega_p)\hat{J}_p^0 + \vec{\mathcal{P}} \cdot \vec{J}$$

$$= \sum_{\mathbf{p},s} 2\lambda_p \hat{U}'^{\dagger} \hat{J}_p^0 \hat{U}'$$

$$\hat{U}' = e^{\sum_p z_p J_p^+} e^{\sum_p \ln(1+|z_p|^2) J_p^0} e^{-\sum_p z_p^* J_p^-}$$

$$z_p = e^{i\delta} \tan\theta_p$$

$$\cos\theta_p = \sqrt{\frac{1}{2}\left(1 + \frac{\omega_c - \omega_p + \mathcal{P}^0}{2\lambda_p}\right)}$$

Spectral Splits

Many people contributed to their explanation

Raffelt, Mirizzi, Dasgupta, Smirnov, Fuller, Qian, Duan, Carlson$\cdots$

$$\begin{aligned}\alpha_1(\mathbf{p}, s) &= \hat{U}'^{\dagger} a_1(\mathbf{p}, s) \hat{U}' = \cos\theta_p \; a_1(\mathbf{p}, s) - e^{i\delta} \sin\theta_p \; a_2(\mathbf{p}, s) \\ \alpha_2(\mathbf{p}, s) &= \hat{U}'^{\dagger} a_2(\mathbf{p}, s) \hat{U}' = e^{-i\delta} \sin\theta_p \; a_1(\mathbf{p}, s) + \cos\theta_p \; a_2(\mathbf{p}, s)\end{aligned}$$

$$\hat{H}^{\mathrm{RPA}} + \omega_c \hat{J}^0 = \sum_{\mathbf{p},s} \lambda_p \left(\alpha_1^{\dagger}(\mathbf{p}, s) \alpha_1(\mathbf{p}, s) - \alpha_2^{\dagger}(\mathbf{p}, s) \alpha_2(\mathbf{p}, s) \right)$$

Spectral Splits

Assume that initially ($\mu \to \infty$) there are more ν_e's and all neutrinos are in flavor eigenstates:

$$\cos\theta_p = \sqrt{\frac{1}{2}\left(1 + \frac{P^0}{|\vec{P}|}\cos 2\theta\right)} \to_{\lim \mu\to\infty} \cos\theta$$

$$\alpha_1(\mathbf{p}, s) = \hat{U}^\dagger a_1(\mathbf{p}, s)\hat{U} \Rightarrow a_e(\mathbf{p}, s)$$

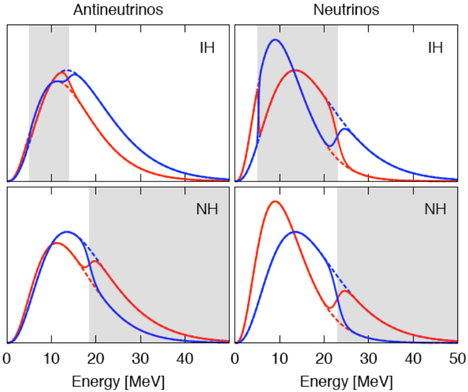
At the end ($\mu \to 0$)

$$\cos\theta_p = \sqrt{\frac{1}{2}\left(1 + \frac{\omega_c - \omega_p}{|\omega_c - \omega_p|}\right)} \Rightarrow \begin{cases} 1 & \omega_p < \omega_c \\ 0 & \omega_p > \omega_c \end{cases}$$

$$\alpha_1(\mathbf{p}, s) = \hat{U}^\dagger a_1(\mathbf{p}, s)\hat{U} \Rightarrow a_1(\mathbf{p}, s)$$

Spectral Splits

from Dasgupta *et al.*



Conclusions

Conclusions

- We examined the many-neutrino gas both from the exact many-body perspective and from the point of view of an effective one-body description formulated with the application of the RPA method. In the limit of the single angle approximation, both the many-body and the RPA pictures possess many constants of motion manifesting the existence of associated dynamical symmetries in the system.
- The existence of constants of motion offer practical ways of extracting information even from exceedingly complex systems. Even when the symmetries which guarantee their existence is broken, they usually provide a convenient set of variables which behave in a relatively simple manner depending on how drastic the symmetry breaking factor is.

Conclusions

Conclusions - continued

- The existence of such invariants naturally lead to associated collective modes in neutrino oscillations. However, symmetries alone do not guarantee the stability of such collective behavior. An extensive numerical study of the collective neutrino phenomena associated with our invariants would shed light on the question of stability.