Lectures on Physical and Mathematical aspects of Gamow States

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Resonances can be defined in several ways, but it is generally accepted that, under rather general conditions, we can associate a resonance to each of the pairs of poles of the analytic continuation of the S-matrix.

In spite of the long-time elapsed since the discovery of alpha decay phenomena in nuclei and their description in terms of resonances, the use of the concept in nuclear structure calculations has been hampered by an apparent contradiction with conventional quantum mechanics, the so-called probability problem. It refers to the fact that a state with complex energy cannot be the eigenstate of a self-adjoint operator, like the Hamiltonian, therefore resonances are not vectors belonging to the conventional Hilbert space.

These lectures are devoted to the description of resonances, i.e. Gamow states, in an amenable mathematical formalism, i.e. Rigged Hilbert Spaces. Since we aim at further applications in the domain of nuclear structure and nuclear many-body problems, we shall address the issue in a physical oriented way, restricting the discussion of mathematical concepts to the needed, unavoidable, background.

From a rigorous historical prospective, the sequence of events and papers leading to our modern view of Gamow vectors in nuclear structure physics includes the following steps:

a) The use of Gamow states in conventional scattering and nuclear structure problems was advocated by Berggren, at Lund, in the 1960s, and by Romo, Gyarmati and Vertse, and by G. Garcia Calderon and Peierls in the late 70s. The notion was recovered years later, by Liotta, at Stockholm, in the 1980s, in connection with the microscopic description of nuclear giant resonances, alpha decay and cluster formation in nuclei. Both at GANIL-Oak Ridge, and at Stockholm, the use of resonant states in shell model calculations was (and still is) reported actively.

b) A parallel mathematical development took place, this time along the line proposed by Bohm, Gadella and collaborators.

Curiously enough both attempts went practically unnoticed to each other for quite a long time until recently, when some of the mathematical and physical difficulties found in numerical applications of Gamow states were discussed on common grounds (Gadella and myself).

Technically speaking, one may summarize the pros and drawbacks ofBerggren and Bohm approaches in the following:

(i) The formalism developed by Berggren is oriented, primarily, to the use of a mixed representation where scattering states and bound states are treated on a foot of equality. In his approach a basis should contain bound states and few resonances, namely those which have a small imaginary part of the energy.

ii) In Bohm’s approach the steps towards the description of Gamow states are based on the fact that the spectrum of the analytically continued Hamiltonian covers the full real axis. As a difference with respect to Berggren’s method, the calculations in Bohm’s approach are facilitated by the use of analytic functions.

The subjects included in these lectures can be ordered in two well-separated conceptual regions, namely:

(a) the S-matrix theory, with reference to Moller wave operators, to the spectral theorem of Gelfand and Maurin, and to the basic notions about Rigged Hilbert Spaces, and

(b) the physical meaning of Gamow resonances in dealing with calculation of observables.

The mathematical tools needed to cover part (a) are presented in a self - contained fashion.

In that part of the lectures
we shall follow, as closely as possible, the discussion advanced in the work of Arno Bohm.

Gamow vectors and decaying states

- Complex eigenvalues
- A model for decaying states
- An explanation and a remedy for the exponential catastrophe
- Gamow vectors and Breit-Wigner distributions

Some references

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-Decay law

Number of particles that have not decay at the time t

$$N_D(t) = N_D(0)e^{-\lambda t}.$$

$$\frac{1}{N_D(0)}\left(-\frac{dN_D(t)}{dt}\right)\Bigg|_{t=0} = \lambda$$ (initial decay rate)

$$N_D(t)/N_D(0) = e^{-\lambda t}$$ (survival)

For a vector that represents the decaying state, like

$$\psi^G(t) = e^{-iHt/\hbar}\psi^G(0)$$

The probability of survival is written

$$|(\psi^G(0), e^{-iHt/\hbar}\psi^G(0))|^2$$

Therefore, this absolute value should be equal to the exponent of the decay rate

The decay condition can be achieved if we think of the state as belonging to a set of states with complex eigenvalues

$$H\psi^G(0) = [E_D - i(\Gamma/2)]\psi^G(0)$$

From where it follows that

$$\begin{aligned} &|(\psi^G(0), e^{-iHt/\hbar}\psi^G(0))|^2 \\ &\quad = |(\psi^G(0), \psi^G(0))e^{-i[E_D - i(\Gamma/2])t/\hbar}|^2 \\ &\quad = e^{-\Gamma t/\hbar}. \end{aligned}$$

The equality between the experimental and theoretical expressions for the probability of survival imply

$$\Gamma = \lambda\hbar.$$

which is, of course, a rather nice but strange relationship which has been obtained under the assumption that there are states, upon which we are acting with of a self-adjoint operator (the Hamiltonian), which possesses complex eigenvalues.

A model for decaying systems

A particle of point-mass (m)
confined in a spherically
symmetric potential well

$$U(r) = \begin{cases} 0, & r < a, \\ U_0, & a \leqslant r \leqslant b, \\ 0, & r > b, \end{cases}$$

The solutions of the non-relativistic Schroedinger equation are written (zero angular momentum for simplicity)

$$i\hbar \frac{\partial \psi(r,\theta,\phi,t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(r,\theta,\phi,t) + U(r)\psi(r,\theta,\phi,t).$$

$$\psi(r,\theta,\phi,t) = [\chi(r)/r] e^{-iEt/\hbar}$$

$$\frac{d^2\chi(r)}{dr^2} = \frac{2m}{\hbar^2} [U(r) - E] \chi(r)$$

The stationary solutions are obtained in the limit of a barrier of infinite height, and they are of the form (inside the barrier)

$$\chi_S(r) = A \sin k_S r$$

with the real wave number

$$k_S = \sqrt{(2m/\hbar^2)E_S}, \quad |A|^2 = 1/2\pi a.$$

$$k_S = n\pi/a, \quad n = 1,2,3,\ldots$$

And for the three regions considered the quasi-stationary solutions are of the form

$$\chi_1(r) = A \sin kr + B \cos kr, \quad 0 \leqslant r \leqslant a,$$
$$\chi_2(r) = Ce^{-Kr} + De^{Kr}, \quad a \leqslant r \leqslant b,$$
$$\chi_3(r) = Fe^{-ikr} + Ge^{ikr}, \quad r > b,$$

the wave-number are real (k) inside and outside the barrier region and complex (K) inside

$$k = \sqrt{\frac{2m}{\hbar^2} E}, \quad K = \sqrt{\frac{2m}{\hbar^2} (U_0 - E)}, \quad E = E_D - i\frac{\Gamma}{2}$$

By requiring the continuity of the function and its first derivative, and by assuming purely outgoing boundary conditions to the solution in the external region, one gets

$$A \sin ka = Ce^{-Ka} + De^{Ka},$$

$$Ak \cos ka = -KCe^{-Ka} + KDe^{ka},$$

$$Ce^{-Kb} + De^{Kb} = Ge^{ikb},$$

$$-KCe^{-Kb} + KDe^{Kb} = ikGe^{ikb}.$$

The solution of the system of equations yields

$$G = [e^{-K(b-a)} e^{-ikb} / (K - ik)] (K \sin ka - k \cos ka) A$$

and the relation between k and K (eigenvalue equation)

$$1 + \frac{K}{k} \tan ka = e^{-2K(b-a)} \frac{K + ik}{K - ik} \left(\frac{K}{k} \tan ka - 1 \right)$$

The initial decay rate can be calculated from the flux of the probability current integrated over the sphere

$$\lambda = \int_{\substack{\text{sphere of}\\ \text{radius } b}} \mathbf{S}(r,\theta,\phi,t=0)\cdot d\,\mathbf{A}$$

$$= \int d\Omega\, b^2 S_r(r=b,\theta,\phi,t=0)$$

where

$$S_r(r=b,\theta,\phi,t=0) = \frac{i\hbar}{2m}\left(\frac{\partial\psi_3^*}{\partial r}\psi_3 - \psi_3^*\frac{\partial\psi_3}{\partial r}\right)_{r=b}$$

(radial component of the current in the external region)

After some trivial calculations one gets (for the external component of the solution)

$$\lambda = (4\pi\hbar k/m)|G|^2.$$

Since the real part of the energy is much smaller that the barrier (but positive) we have

$$k_0 = \sqrt{(2m/\hbar^2)E_D}, \quad K_0 = \sqrt{(2m/\hbar^2)(U_0 - E_D)},$$

Then, by replacing G for its value in terms of the approximated wave numbers, we find

$$\lambda \simeq (4\pi\hbar k_0/m)\left[e^{-2K_0(b-a)}/(K_0^2+k_0^2)\right](k_0 \cos k_0 a)^2$$
$$\times\left[(K_0/k_0)\tan k_0 a - 1\right]^2 |A|^2$$

$$\lambda \simeq (8\hbar k_0^3/maK_0^2)\,e^{-2K_0(b-a)}$$

and

$$\Gamma = \lambda \hbar = (8\hbar^2 k_0^3 / m a K_0^2) e^{-2K_0(b-a)}$$

If we proceed in the same fashion starting from the definition of the Gamow vector, we find for the complex wave number

$$k = \sqrt{\frac{2m}{\hbar^2} E} = \sqrt{\frac{2m}{\hbar^2}\left(E_D - i\frac{\Gamma}{2}\right)}$$

by keeping only leading order terms, since the imaginary part of the energy is much smaller that the real part:

$$k \simeq k_0 - im\Gamma/2\hbar^2 k_0$$

$$K \simeq K_0$$

With these approximations, the relationship between k and K determines the structure of the decay width:

$$1 + K_0 \frac{\tan ka}{k} \simeq e^{-2K_0(b-a)} \frac{(K_0 + ik_0)^2}{K_0^2 + k_0^2} \times \left(\frac{K_0}{k_0} \tan k_0 a - 1 \right).$$

and

$$\frac{\tan ka}{k} \simeq \frac{\tan k_0 a}{k_0} + \left(\frac{a}{k_0 \cos^2 k_0 a} - \frac{\tan k_0 a}{k_0^2} \right) \left(\frac{-im\Gamma}{2\hbar^2 k_0} \right)$$

And by equating the imaginary parts of both equations one gets

$$\frac{1}{k_0}\left(\frac{K_0 a}{\cos^2 k_0 a} - \frac{K_0}{k_0}\tan k_0 a\right)\left(\frac{-im\Gamma}{2\hbar^2 k_0}\right)$$
$$\simeq \frac{2ik_0K_0}{K_0^2 + k_0^2}\left(\frac{K_0}{k_0}\tan k_0 a - 1\right)e^{-2K_0(b-a)}$$

which is fulfilled if (at the same order)

$$\Gamma \simeq (8\hbar^2 k_0^3 / m a K_0^2) e^{-2K_0(b-a)}$$

This last result is quite remarkable, because it shows that decaying states can indeed be represented by states with complex eigenvalues. It also leads to the rather natural interpretation of the decay time as the inverse of the imaginary part of the complex energy.

The exponential catastrophe

For well behaved vectors in Hilbert space one has the probability density (finite and normalizable)

$$\rho(\mathbf{x}) = |\langle \mathbf{x}|\phi\rangle|^2$$

$$\int d^3x\, \rho(\mathbf{x}) < \infty$$

But not all vectors are proper vectors, since (for instance for Dirac's kets)

And it means that, for them

$$\langle \mathbf{x} | \mathbf{p} \rangle = (2\pi\hbar)^{-3/2} e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}$$

$$\rho(\mathbf{x}) = |\langle \mathbf{x} | \mathbf{p} \rangle|^2 = (2\pi\hbar)^{-3}$$

Although it is finite at each point (in space) it is infinite when integrated over the complete domain. For a Gamow vector one has, similarly:

$$\rho(\mathbf{x},t) = |\langle r,\theta,\phi | \psi^G(t) \rangle|^2 = |\psi^G(r,\theta,\phi,t)|^2$$

from the wave function in the external region one has

$$\chi_3(r,\theta,\phi,t) = \chi_3(r,t) \simeq G e^{i(k_0 r - E_D t/\hbar)} \times e^{-(\Gamma/2)(t - mr/\hbar k_0)/\hbar}, \quad r > b.$$

one has

$$\rho(r,t) \simeq (|G|^2/r^2)\, e^{-\Gamma(t - mr/\hbar k_0)/\hbar}, \quad r > b.$$

The factor between parenthesis can be writen in terms of a "initial time" associated to the approximated wave-number, leading to:

$$\rho(r,t) \simeq (|G|^2/r^2)e^{-\Gamma[t-t_0(r)]/\hbar},$$

$$r > b, \quad t > t_0(r)$$

This result (which is valid for times t greater than the "initial " time) means that a detector place at a distance r detects a counting rate that is maximum at the initial time and decreases exponentially with increasing time (Bohm-Gadella-Mainland)

Thus, when correctly interpreted, Gamow vectors leads to an exponential decay law and not to an exponential catastrophe.

Gamow vectors and Breit-Wigner distributions

At this point (and as a first contact with the proper mathematics) we shall introduce another tool to understand Gamow vectors. That is the analogous of Dirac's spectral theorem and the nuclear spectral theorem (also called Maurin-Gelfand-Vilenkin theorem). This will be our first exposure to the concept of Rigged Hilbert Spaces (or Gelfand's triads).

Dirac spectral theorem

In the infinite dimensional Hilbert space we can define an orthonormal system of eigenvectors of an operator with a discrete set of eigenvalues (the Hamiltonian operator, for instance), such that

$$H\,|E_n) = E_n\,|E_n), \quad E_n = E_0, E_1, E_2 \ldots,$$

Then, every vector of the same space can be expanded as:

$$\varphi = \sum_{n=0,1,\ldots}^{\infty} |E_n)(E_n|\varphi)$$

In scattering problems the Hamiltonian has a continuous spectrum (in addition to bound states), and this part of the spectrum can be added to the expansion

$$\varphi = \sum_{n=0,1,\ldots}^{\infty} |E_n)(E_n|\varphi) + \int_0^{\infty} dE\,|E^-\rangle\langle^-E|\varphi\rangle$$

$$H|E^-\rangle = E|E^-\rangle, \quad 0 \leqslant E \leqslant \infty$$

To show that this is indeed a correct expansion we shall introduce, in addition to the Hilbert space (of normalizable, integrable functions) a subspace of all integrable functions in the energy representation. We think of these functions as " well behaved" (infinitely differentiable and rapidly decreasing in the energy rep) (technically speaking they are elements of the Schwartz space).

We also need another space, which should accomodate the antilinear mappings of these functions (something is antilinear if F(ax+by)=a*F(x)+b*F(y)).

We then have:

a)The Hilbert space of regular vectors $\mathscr{H}$.

b)The space (Schawrtz) of test functions $\Phi \subset \mathscr{H}$

c)Its antilinear mapping. $\Phi^{\times}$

Under these conditions, Gelfand defines the triads (or Rigged Hilbert Spaces, or triplets) as the construction

$$\Phi \subset \mathscr{H} \subset \Phi^{\times}$$

Nuclear spectral theorem:

Let H be a self-adjoint operator in the Hilbert space. Then it exists in the subspace $\Phi^{\times}$

a set of elements with the property

$$\langle H\psi|E^{-}\rangle = E\,\langle\psi|E^{-}\rangle, \quad \text{for every } \psi\in\Phi$$

Such that any function of (b) can be written in the form

$$\varphi = \sum_{n=0,1,\ldots}^{\infty} |E_n)(E_n|\varphi) + \int_0^{\infty} dE\,|E^{-}\rangle\langle{}^{-}E|\varphi\rangle$$

Let us see how these notions apply in the case of Gamow vectors

A particular generalized eigenvector with complex eigenvalues can be defined by:

$$\psi^G = \int_{-\infty}^{\infty} dE \, |E^-\rangle \frac{1}{i} \sqrt{\frac{\Gamma}{2\pi}} \frac{1}{E_D - i(\Gamma/2) - E}.$$

which is a functional over well behaved functions

$$\langle \chi | \psi^G \rangle = \int_{-\infty}^{\infty} dE \, \langle \chi | E^- \rangle \frac{1}{i} \sqrt{\frac{\Gamma}{2\pi}} \frac{1}{E_D - i(\Gamma/2) - E}$$

Properties and conditions on these vectors:

1) (Generalized eigenvalue equation)

$$\langle\chi|H|\psi^G\rangle = [E_D - (i/2)\Gamma]\langle\chi|\psi^G\rangle$$

2) (Time evolution)

$$\begin{aligned}\langle\chi|\psi^G(t)\rangle &= \langle\chi|e^{-iHt/\hbar}|\psi^G\rangle \\ &= e^{-iE_Dt/\hbar - \Gamma t/2\hbar}\langle\chi|\psi^G\rangle, \quad t \geqslant 0\end{aligned}$$

3) (Probability of detecting a given value of the energy E)

$$\begin{aligned}P(E) &= |\langle E^-|\psi^G\rangle|^2 \\ &= (\Gamma/2\pi)\{1/[(E-E_D)^2 + (\Gamma/2)^2]\}\end{aligned}$$

Proof of the properties

$$
\begin{aligned}
&\langle \chi | H | \psi^G \rangle \\
&= \int_{-\infty}^{\infty} dE \, \langle \chi | H | E^- \rangle \frac{1}{i} \sqrt{\frac{\Gamma}{2\pi}} \frac{1}{E_D - i(\Gamma/2) - E} \\
&= \int_{-\infty}^{\infty} dE \, E \, \langle \chi | E^- \rangle \frac{1}{i} \sqrt{\frac{\Gamma}{2\pi}} \frac{1}{E_D - i(\Gamma/2) - E},
\end{aligned}
$$

$$
\begin{aligned}
&\langle \chi | e^{-iHt/\hbar} | \psi^G \rangle \\
&\quad = \int_{-\infty}^{\infty} dE \, \langle \chi | e^{-iHt/\hbar} | E^- \rangle \frac{1}{i} \\
&\qquad \times \sqrt{\frac{\Gamma}{2\pi}} \frac{1}{E_D - i(\Gamma/2) - E} \\
&\quad = \int_{-\infty}^{\infty} dE \, e^{-iEt/\hbar} \langle \chi | E^- \rangle \frac{1}{i} \\
&\qquad \times \sqrt{\frac{\Gamma}{2\pi}} \frac{1}{E_D - i(\Gamma/2) - E}.
\end{aligned}
$$

(use Cauchy integral formula):

$$\oint dz \frac{f(z)}{z-a} = 2\pi i f(a),$$

The integral is calculated over a closed contour (real energy axis) and an infinite semicircle in the lower half of the complex plane, leading to

$$\langle \chi | \psi^G \rangle = \sqrt{2\pi\Gamma} \; \langle \chi | E_D - i(\Gamma/2 -) \rangle.$$

$$\langle \chi | H | \psi^G \rangle = (E_D - i(\Gamma/2)) \sqrt{2\pi\Gamma} \; \langle \chi | E_D - i(\Gamma/2 - \rangle.$$

$$\langle\chi|e^{-iHt/\hbar}|\psi^G\rangle = e^{-iE_Dt/\hbar}e^{-\Gamma t/2\hbar} \times\sqrt{2\pi\Gamma}\ \langle\chi|E_D - i(\Gamma/2-\rangle, \quad t\geqslant 0.$$

since $$e^{-iEt/\hbar}\langle\chi|E^-\rangle$$

vanishes on the infinite semicircle in the lower half of the complex plane only for t >0.

More generally, since there is one (pair of) S-matrix pole at E, then every well behaved vector can be given by the generalized expansion (A. Mondragon has shown it first, and then with E. Hernandez) we shall come back to this result later on)

$$\varphi = \int_0^{-\infty} dE\, S_{11}(E)|E\rangle\langle E|\varphi\rangle + \psi^G\langle\psi^G|\varphi\rangle$$

$$= \varphi^{\text{background}} + \psi^G\langle\psi^G|\varphi\rangle,$$

The approximation to the exponential decay law

- The exponential decay law is a consequence of the choice of the Breit-Wigner energy distribution (a natural choice, indeed).
- Influence of the direct integration over the physically positive energy spectrum: it affects the decay law

$$\omega(E)=\frac{\Gamma}{\pi[(E-E_0)^2+\Gamma^2]}$$

It seems natural to think of the Breit-Wigner energy distribution as the result of the composition between to branches, like:

$$\omega(E)=\frac{\Gamma}{[E-(E_0-i\Gamma)][E-(E_0+i\Gamma)]}\cdot f(E)$$

Where the function f(E) has no poles (Krylov) to ensure that the only poles are those at the positive and negative half-planes.

Then, if we integrate a given amplitude, like

$$A(t)=\int_0^{\infty} e^{-iEt}\omega(E)dE$$

in the real axis, and use Cauchy formula, we get, for the relevant part of the integral

$$A(t)\approx\int_{-\infty}^{\infty} e^{-iEt}\,\frac{\Gamma}{\pi[(E-E_0)^2+\Gamma^2]}\,dE$$

that the main contribution comes from the pole located at the lower half-plane, at it yields:

$$A(t)=e^{-iE_0t}\cdot e^{-\Gamma t}$$

this result can be generalized easily by considering a semibounded spectrum (that is : the real positive set of eigenvalues)

$$A_{kl}(t) = \frac{1}{\pi}\int_0^\infty \frac{a_{kl}e^{-iE_k t}}{[(E-E_k)^2+\Gamma_k^2]^{l+1}}\,dE$$

$$= \frac{1}{\pi}\int_0^\infty e^{-iE_j t}\frac{a_{kl}\cos(ut)}{[u^2+\Gamma_k^2]^{l+1}}\,du$$

$$-\frac{i}{\pi}\int_0^\infty e^{-iE_k t}\frac{a_{kl}\sin(ut)}{(u^2+\Gamma_k^2)^{j+1}}$$

$$+\frac{1}{\pi}\int_{-E_k}^0 e^{-iE_k t}\frac{a_{kl}e^{-iut}}{(u^2+\Gamma_k^2)^{l+1}}\,du$$

$$= \frac{1}{\sqrt{\pi}}a_{kl}e^{-iEt}\left(\frac{t}{2\Gamma_k}\right)^{l+1/2} K_{l+1/2}(\Gamma_k t)/\Gamma(l+1)$$

$$-\frac{ia_{kl}}{\pi t[E_k^2+\Gamma_k^2]^{l+1}} - \frac{2a_{kl}E_k}{\pi t^2(E_k^2+\Gamma_k^2)^{l+2}}\cdots,$$

where the functions K are modified Bessel functions of imaginary argument. The term by term expansion of A(t) reads

$$A_{00}(t)=a_{00}e^{-iE_0t}e^{-\Gamma_0t}\cdot\frac{1}{2\Gamma_0}-\frac{ia_{00}}{\pi t(E_0^2+\Gamma_0^2)}-\frac{2a_{00}E_0}{\pi t^2(E_0^2+\Gamma_0^2)^2},$$

$$A_{11}(t)=a_{11}e^{-iE_1t}e^{-\Gamma_1t}(1+\Gamma_1t)\frac{1}{4\Gamma_1^3}-\frac{ia_{11}}{\pi t(E_1^2+\Gamma_1^2)^2}-\frac{2a_{11}E_1}{\pi t^2(E_1^2+\Gamma_1^2)^3}$$

$$A_{12}(t)=a_{12}e^{-iE_2t}e^{-\Gamma_2t}(1+\Gamma_2t)\frac{1}{4\Gamma_2^3}-\frac{ia_{12}}{\pi t(E_2^2+\Gamma_2^2)^2}-\frac{2a_{12}E_2}{\pi t^2(E_2^2+\Gamma_2^2)^3}$$

Then, the first term already gives the expected dependence (decay law) as a function of the imaginary part of the complex energy, and it is valid provided $t \gg 1/\text{Real part of } (E)$.

The same result can be obtained by applying the steepest descent method, by re-writting A(t) as

$$A_0(t)=\frac{1}{\pi}\Gamma t\int_0^\infty d\sigma\int_0^\infty dE\ e^{-iEt}e^{-\sigma t[(E-E_0)^2+\Gamma^2]}$$

$$=\frac{1}{\pi}\Gamma t\int_0^\infty d\sigma\int_0^\infty e^{tf(z)}g(z)dz,$$

where

$$f(z)=-iz-\sigma[(z-E_0)^2+\Gamma^2]g(z)=1,$$

$$f'(z)=-i-2\sigma(z-E_0)=0 \quad \text{for } z_0=E_0-\frac{i}{2\sigma}$$

$$f''(z)=-2\sigma.$$

then

$$\int_0^\infty e^{tf(z)} g(z)dz \simeq \frac{\sqrt{2\pi} g(z_0)}{\sqrt{t|f''(z_0)|}} e^{tf(z_0)} e^{i\alpha}$$

and

$$= \frac{\sqrt{2\pi}}{\sqrt{2\sigma t}} e^{t(-iE_0 - 1/2\sigma + 1/4\sigma - \sigma\Gamma^2)},$$

$$A_0(t) = \frac{1}{\pi} \Gamma t \int_0^\infty \sqrt{\frac{\pi}{t\sigma}} e^{t(-iE_0 - 1/4\sigma - \sigma\Gamma^2)},$$

$$= \Gamma \sqrt{\frac{t}{\pi}} e^{-iE_0 t} \int_0^\infty d\sigma \frac{e^{-t(1/4\Gamma^2\sigma + 1/4\sigma)}}{\sqrt{\sigma}}$$

$$p = 2\sigma\Gamma, \quad \sigma = \frac{p}{2\Gamma}, \quad dp = 2\Gamma \, d\sigma.$$

from which

$$A_0(t) = \Gamma \sqrt{\frac{t}{\pi}}\, e^{-iE_0 t} \int_0^\infty \frac{1}{\sqrt{2\Gamma}}\, dP\, \frac{e^{-t(\Gamma/2P + \Gamma P/2)}}{\sqrt{P}}$$

$$= \Gamma \sqrt{\frac{t}{\pi}}\, \frac{e^{-iE_0 t}}{\sqrt{2\Gamma}} \int_0^\infty dP\, \frac{e^{-\Gamma t/2(P+1/P)}}{\sqrt{P}}$$

$$= \Gamma \sqrt{\frac{t}{\pi}}\, \frac{e^{-iE_0 t}}{\sqrt{2\Gamma}}\, K_{1/2}(\Gamma t),$$

and with the modified Bessel functions

$$K_\nu(x) = \int_0^\infty ds\, \frac{e^{-x/2(s+1/s)}}{S^{1-\nu}},$$

$$K_\nu(x) = \frac{\pi}{2\sin\nu\pi}\{I_{-\nu}(x) - I_\nu(x)\}$$

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos hx, \quad I_{1/2} = \sqrt{\frac{2}{\pi x}}\sin hx,$$

$$K_{1/2}(x) = \frac{\pi}{2}\sqrt{\frac{2}{\pi x}}\,e^{-x} = \sqrt{\frac{\pi}{2x}}\,e^{-x},$$

finally

$$A_0(t)=\Gamma\sqrt{\frac{t}{\pi}}\frac{e^{-iE_0t}}{\sqrt{2\Gamma}}K_{1/2}(\Gamma t)=\frac{1}{2}e^{-iE_0t}e^{-\Gamma t}$$

which is, again, the expected time dependence of the amplitude $A(t)$

Gamow vectors: a tour from the elementary concepts to the (more elaborate) mathematical concepts

Some references:

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-We shall compare the description of Gamow vectors

1)in the scattering problem,

2)the S-matrix formalism, and

3)in the rigged Hilbert space framework

All are equivalent way of introducing Gamow vectors and their properties, but some are more illustrative than others, as we shall see in the following notes.

-Resonances for a square barrier potential

Since we have already discussed the case in the first lecture, we shall review here the main results and expressions for a later use. We shall start with Schroedinger equation in the Dirac notation. When the energies belong to the continuum spectrum of H, the vectors are Dirac's kets and they are not vectors in the Hilbert space (they are not square integrable). However, we can use them to expand the space of wave functions

$$\varphi=\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_{0}^{\infty} dE|E,l,m\rangle\langle E,l,m|\varphi\rangle.$$

We are searching for solutions of the equation

$$\langle \mathbf{x}|H|E\rangle = \left(\frac{-\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right)\langle \mathbf{x}|E\rangle = E\langle \mathbf{x}|E\rangle$$

$$V(\mathbf{x}) = V(r) = \begin{cases} 0, & 0<r<a \\ V_0, & a<r<b \\ 0, & b<r<\infty . \end{cases}$$

in spherical coordinates it reads

$$\langle r,\theta,\phi|H|E,l,m\rangle = \left(\frac{-\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right)$$
$$\times\langle r,\theta,\phi|E,l,m\rangle$$

with

$$\langle r,\theta,\phi|E,l,m\rangle \equiv \langle r|E\rangle_l\langle\theta,\phi|l,m\rangle$$
$$\equiv \frac{1}{r}\chi_l(r;E)Y_{l,m}(\theta,\phi)$$

The solutions are

$$\chi(r;E)=\begin{cases}\alpha_1 e^{ikr}+\beta_1 e^{-ikr}, & 0<r<a\\ \alpha_2 e^{iQr}+\beta_2 e^{-iQr}, & a<r<b\\ \mathcal{F}_1 e^{ikr}+\mathcal{F}_2 e^{-ikr}, & b<r<\infty\end{cases}$$

$$k=\sqrt{\frac{2m}{\hbar^2}E}$$

$$Q=\sqrt{k^2-\frac{2m}{\hbar^2}V_0}=\sqrt{\frac{2m}{\hbar^2}(E-V_0)}.$$

Boundary and continuity conditions

$$\chi(0;E)=0,$$

$$\chi(a-0;E)=\chi(a+0;E),$$

$$\chi'(a-0;E)=\chi'(a+0;E),$$

$$\chi(b-0;E)=\chi(b+0;E),$$

$$\chi'(b-0;E)=\chi'(b+0;E),$$

$$|\chi(r;E)|<\infty.$$

they are of the form

$$\alpha_2 e^{iQa} + \beta_2 e^{-iQa} = \alpha \sin(ka),$$

$$iQ(\alpha_2 e^{iQa} - \beta_2 e^{-iQa}) = \alpha k \cos(ka),$$

$$\mathcal{F}_1 e^{ikb} + \mathcal{F}_2 e^{-ikb} = \alpha_2 e^{iQb} + \beta_2 e^{-iQb},$$

$$ik(\mathcal{F}_1 e^{ikb} - \mathcal{F}_2 e^{-ikb}) = iQ(\alpha_2 e^{iQb} - \beta_2 e^{-iQb})$$

with solutions

$$\alpha_2(k) = \frac{1}{2} e^{-iQa} \left[\sin(ka) + \frac{k}{iQ} \cos(ka) \right] \alpha(k),$$

$$\beta_2(k) = \frac{1}{2} e^{iQa} \left[\sin(ka) - \frac{k}{iQ} \cos(ka) \right] \alpha(k),$$

$$\begin{aligned} \mathcal{F}_1(k) = \frac{e^{-ikb}}{4} \Bigg[& \left(1 + \frac{Q}{k} \right) e^{iQ(b-a)} \left(\sin(ka) + \frac{k}{iQ} \cos(ka) \right) \\ & + \left(1 - \frac{Q}{k} \right) e^{-iQ(b-a)} \\ & \times \left(\sin(ka) - \frac{k}{iQ} \cos(ka) \right) \Bigg] \alpha(k) \end{aligned}$$

$$
\begin{aligned}
\mathcal{F}_2(k) = \frac{e^{ikb}}{4}\Bigg[&\left(1 - \frac{Q}{k}\right) e^{iQ(b-a)} \left(\sin(ka) + \frac{k}{iQ}\cos(ka) \right) \\
&+ \left(1 + \frac{Q}{k}\right) e^{-iQ(b-a)} \\
&\times \left(\sin(ka) - \frac{k}{iQ}\cos(ka) \right) \Bigg] \alpha(k).
\end{aligned}
$$

Thus, the zero angular momentum solution reads

$$\langle r,\theta,\phi|E\rangle = \frac{\chi(r;E)}{r} Y_{0,0}(\theta,\phi) = \frac{\chi(r;E)}{r} \sqrt{\frac{1}{4\pi}}$$

$$0 \leqslant E < \infty$$

with

$$r\langle r|E\rangle_{l=0} = \chi(r;E)$$

$$= \begin{cases} \alpha(k)\sin(kr), & 0<r<a \\ \alpha_2(k)e^{iQr} + \beta_2(k)e^{-iQr}, & a<r<b \\ \mathcal{F}_1(k)e^{ikr} + \mathcal{F}_2(k)e^{-ikr}, & b<r<\infty \end{cases} .$$

the expansion of the wave function will then be

$$\varphi(r,\theta,\phi)=\int_0^\infty dE\,\frac{\chi(r;E)}{r}\,Y_{0,0}(\theta,\phi)\,\varphi(E)$$

and from it one has (in bra and kets notation)

$$\langle r,\theta,\phi|\varphi\rangle=\int_0^\infty dE\langle r,\theta,\phi|E\rangle\langle E|\varphi\rangle$$

S-matrix

$$\langle {}^{-}E,l,m|E',l',m'^{+}\rangle = S_l(E)\,\delta(E-E')\,\delta_{l,l'}\,\delta_{m,m'}$$

$$\langle r|E^{+}\rangle = \frac{-1}{2i}\,\frac{\chi(r;E)}{\mathcal{F}_2}$$

$$\langle r|E^{-}\rangle = \frac{1}{2i}\,\frac{\chi(r;E)}{\mathcal{F}_1}.$$

the probability-amplitud of detecting an out-state (superscipt -) in an in-state (superscript +) is

$$(\psi^-,\varphi^+)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^{\infty}dE\langle\psi^-|E,l,m^-\rangle S_l(E)$$
$$\times\langle^+E,l,m|\varphi^+\rangle.$$

the S-matrix, in the energy representation, is written

$$S(E) \equiv S(k) = -\frac{\mathcal{F}_1(k)}{\mathcal{F}_2(k)}$$

the continuation of S is analytic except at its poles, which are determined by

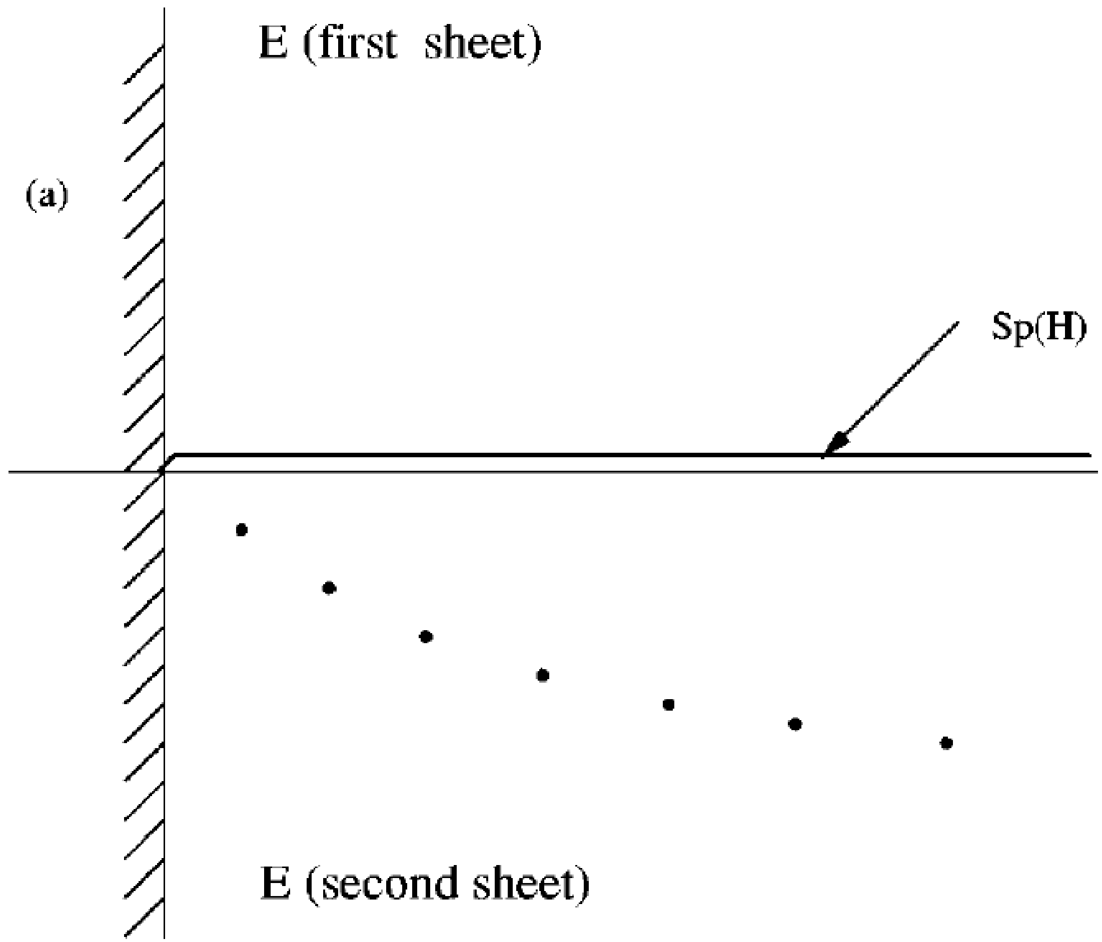
$$\mathcal{F}_2(k) = 0,$$

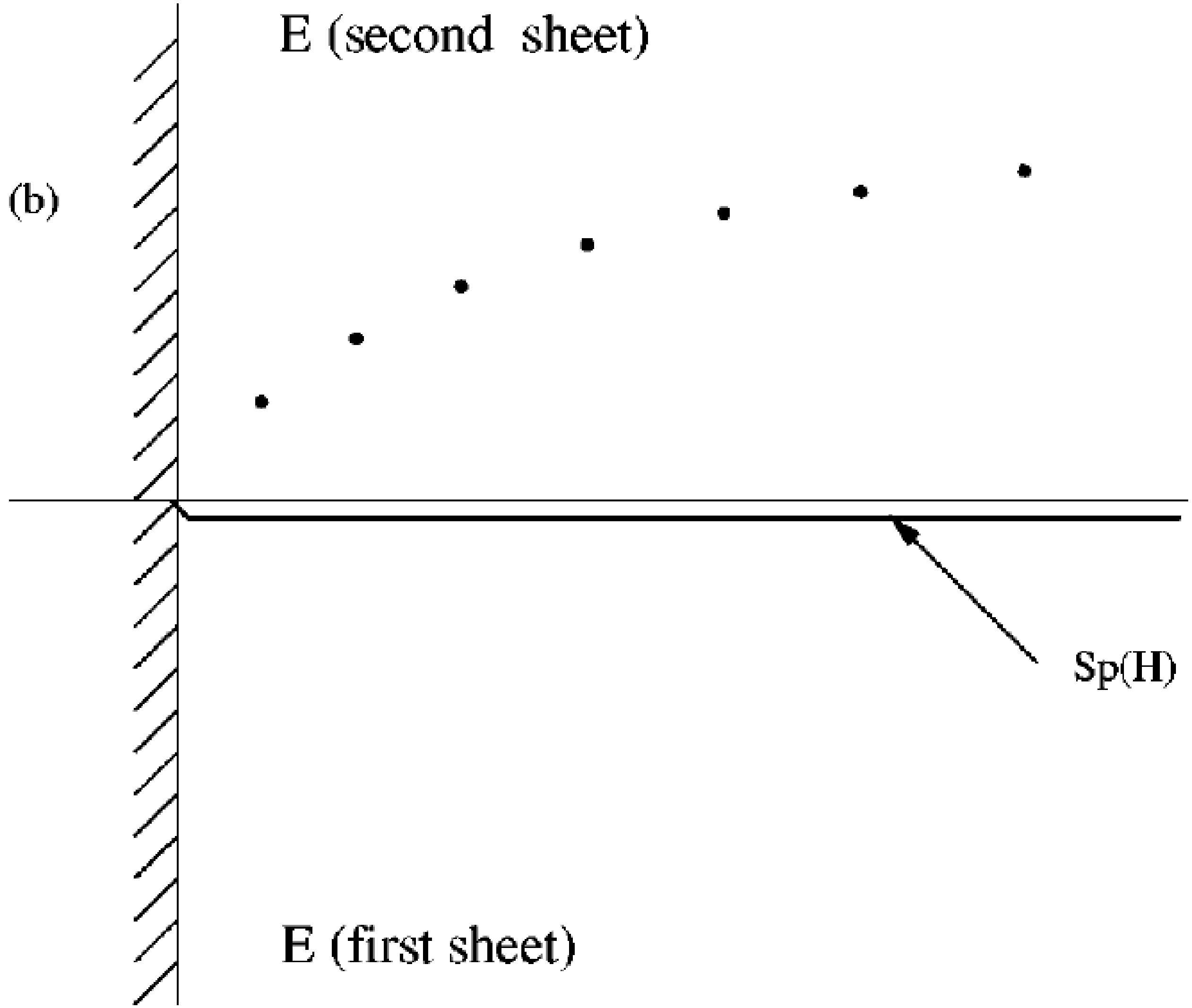
that is

$$\left(1-\frac{Q}{k}\right)e^{iQ(b-a)}\left[\sin(ka)+\frac{k}{iQ}\cos(ka)\right] \\ +\left(1+\frac{Q}{k}\right)e^{-iQ(b-a)}\left[\sin(ka)-\frac{k}{iQ}\cos(ka)\right]=0$$

the solutions come in pairs of complex conjugate values of the energy,

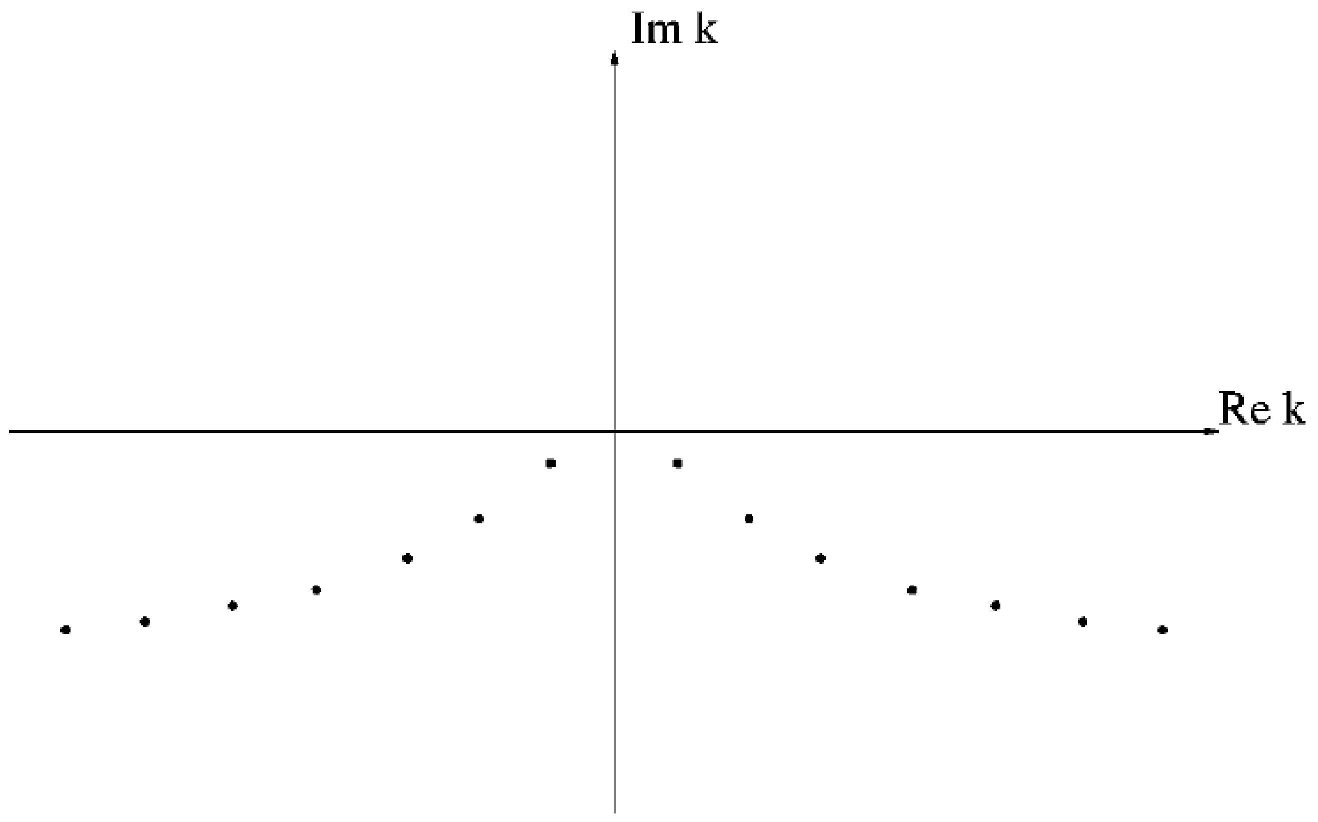
Decaying resonance energies (square well potential)





Growing resonance energies (square well potential)

-resonance wave numbers (for the square well potential)



Gamow vectors

The time independent Schroedinger equation is written

$$H|z_R\rangle = z_R|z_R\rangle$$

$$\langle \mathbf{x}|H|z_R\rangle = z_R\langle \mathbf{x}|z_R\rangle$$

and the radial part of the zero angular momentum Gamow vector is given by

$$\chi(r;z_R) = \begin{cases} \alpha_1 e^{ikr} + \beta_1 e^{-ikr}, & 0<r<a \\ \alpha_2 e^{iQr} + \beta_2 e^{-iQr}, & a<r<b \\ \mathcal{F}_1 e^{ikr} + \mathcal{F}_2 e^{-ikr}, & b<r<\infty \end{cases}$$

with the complex wave number

$$k = \sqrt{\frac{2m}{\hbar^2} z_R}$$

and

$$Q = \sqrt{k^2 - \frac{2m}{\hbar^2} V_0} = \sqrt{\frac{2m}{\hbar^2} (z_R - V_0)}$$

the boundary conditions are:

$$\chi(0;z_R)=0,$$

$$\chi(a-0;z_R)=\chi(a+0;z_R),$$

$$\chi'(a-0;z_R)=\chi'(a+0;z_R),$$

$$\chi(b-0;z_R)=\chi(b+0;z_R),$$

$$\chi'(b-0;z_R)=\chi'(b+0;z_R),$$

$$\chi(r;z_R)\sim e^{ikr}, \quad r\longrightarrow\infty.$$

notice the purely outgoing condition at infinity

the purerly outgoing boundary condition
(impossed to Gamow vectors) is often written

$$\lim_{r \to \infty} \frac{d\chi(r;z_R)}{dr} - ik\chi(r;z_R) = 0.$$

next, we have to solve the equations to calculate the radial wave functions explicitely, in matrix notations they read

$$
\begin{pmatrix} \sin(ka) & 0 & -e^{iQa} & -e^{-iQa} \\ k\cos(ka) & 0 & -iQe^{iQa} & iQe^{-iQa} \\ 0 & e^{ikb} & -e^{iQb} & -e^{-iQb} \\ 0 & ike^{ikb} & -iQe^{iQb} & iQe^{-iQb} \end{pmatrix} \begin{pmatrix} \alpha \\ \mathcal{F}_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
$$

where $\alpha = 2i\alpha_1$

the determinant of the system is then written

$$\begin{vmatrix} \sin(ka) & 0 & -e^{iQa} & -e^{-iQa} \\ k\cos(ka) & 0 & -iQe^{iQa} & iQe^{-iQa} \\ 0 & e^{ikb} & -e^{iQb} & -e^{-iQb} \\ 0 & ike^{ikb} & -iQe^{iQb} & iQe^{-iQb} \end{vmatrix} = 0$$

leading to the dispersion relation

$$\left(1-\frac{Q}{k}\right)e^{iQ(b-a)}\left[\sin(ka)+\frac{k}{iQ}\cos(ka)\right]+\left(1+\frac{Q}{k}\right)$$
$$\times e^{-iQ(b-a)}\left[\sin(ka)-\frac{k}{iQ}\cos(ka)\right]=0.$$

Then for each pair of complex energies we find the decaying

$$\chi^{\text{decaying}}(r;z_R)=\begin{cases}\sin(k_d r), & 0<r<a\\ \alpha_2(k_d)e^{iQ_d r}+\beta_2(k_d)e^{-iQ_d r}, & a<r<b\\ \mathcal{F}_1(k_d)e^{ik_d r}, & b<r<\infty\,,\end{cases}$$

$$k_d=\sqrt{2m/\hbar^2(E_R-i\Gamma_R/2)}$$

$$Q_d^2=k_d^2-2m/\hbar^2 V_0\,.$$

and growing radial solutions

$$\chi^{\text{growing}}(r;z_R^*) = \begin{cases} \sin(k_g r), & 0<r<a \\ \alpha_2(k_g)e^{iQ_g r}+\beta_2(k_g)e^{-iQ_g r}, & a<r<b \\ \mathcal{F}_1(k_g)e^{ik_g r}, & b<r<\infty \end{cases},$$

$$k_g = \sqrt{2m/\hbar^2(E_R + i\Gamma_R/2)}$$

$$Q_g^2 = k_g^2 - 2m/\hbar^2 V_0$$

Green function method

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}+V(r)-E\right)G(r,r';E)=-\delta(r-r'),$$

the formal solutions are

$$G(r,r';E)=\frac{2m}{\hbar^2}\frac{\chi(r_<;E)\psi(_>;E)}{W(\chi,\psi)}$$

where the functions entering the definition of G are the solutions of the radial equation which vanishes at the origin

$$\chi(r_< ;E)$$

or fulfills a purely outgoing boundary condition

$$\psi(_> ;E)$$

and their wronskian

$$W(\chi,\psi)$$

$$\psi(a-0;E)=\psi(a+0;E),$$

$$\psi'(a-0;E)=\psi'(a+0;E),$$

$$\psi(b-0;E)=\psi(b+0;E),$$

$$\psi'(b-0;E)=\psi'(b+0;E),$$

$$\psi(r;E)\sim e^{ikr},\quad r\to\infty.$$

The boundary conditions for the outgoing wave are

$$\psi(a-0;E)=\psi(a+0;E),$$

$$\psi'(a-0;E)=\psi'(a+0;E),$$

$$\psi(b-0;E)=\psi(b+0;E),$$

$$\psi'(b-0;E)=\psi'(b+0;E),$$

$$\psi(r;E)\sim e^{ikr}, \quad r\longrightarrow\infty.$$

With solutions given by

$$\psi(r;k)=\begin{cases} a_1(k)e^{ikr}+b_1(k)e^{-ikr}, & 0<r<a \\ a_2(k)e^{iQr}+b_2(k)e^{-iQr}, & a<r<b \\ e^{ikr}, \quad b<r<\infty \ , \end{cases}$$

the wronskian is of the form

$$W(\chi,\psi)(k)=\chi(r)\psi'(r)-\chi'(r)\psi(r)=2ik\mathcal{F}_2(k)$$

the structure of the Green function is, therefore

$$G(r,r';k)=\frac{2m}{\hbar^2}\frac{\chi(r_<;k)\psi(r_>;k)}{2ik\mathcal{F}_2(k)}.$$

With residues at the poles (which are the same poles of S)

$$\mathrm{res}[G(r,r';k)]_{k=k_d}=\frac{2m}{\hbar^2}\frac{1}{2ik_d\mathcal{F}_1(k_d)\mathcal{F}_2'(k_d)}\chi^{\mathrm{decaying}}(r_<;k_d)\chi^{\mathrm{decaying}}(r_>;k_d).$$

Complex vector expansions

From the amplitude

$$(\psi^-,\varphi^+)=\int_0^\infty \langle\psi^-|E^-\rangle S(E)\langle E^+|\varphi^+\rangle dE$$

We can extract the contribution of resonances by making an analytic continuation of the S-matrix and by deforming the contour of integration

$$(\psi^-, \varphi^+) = \int_0^{-\infty} \langle \psi^- | E^+ \rangle \langle E^+ | \varphi^+ \rangle dE$$

$$-2\pi i \sum_{n=0}^{\infty} r_n \langle \psi^- | z_{d,n}^- \rangle \langle {}^+ z_{d,n} | \varphi^+ \rangle$$

$$z_{d,n} = E_n - i\Gamma_n/2$$

From it, one gets

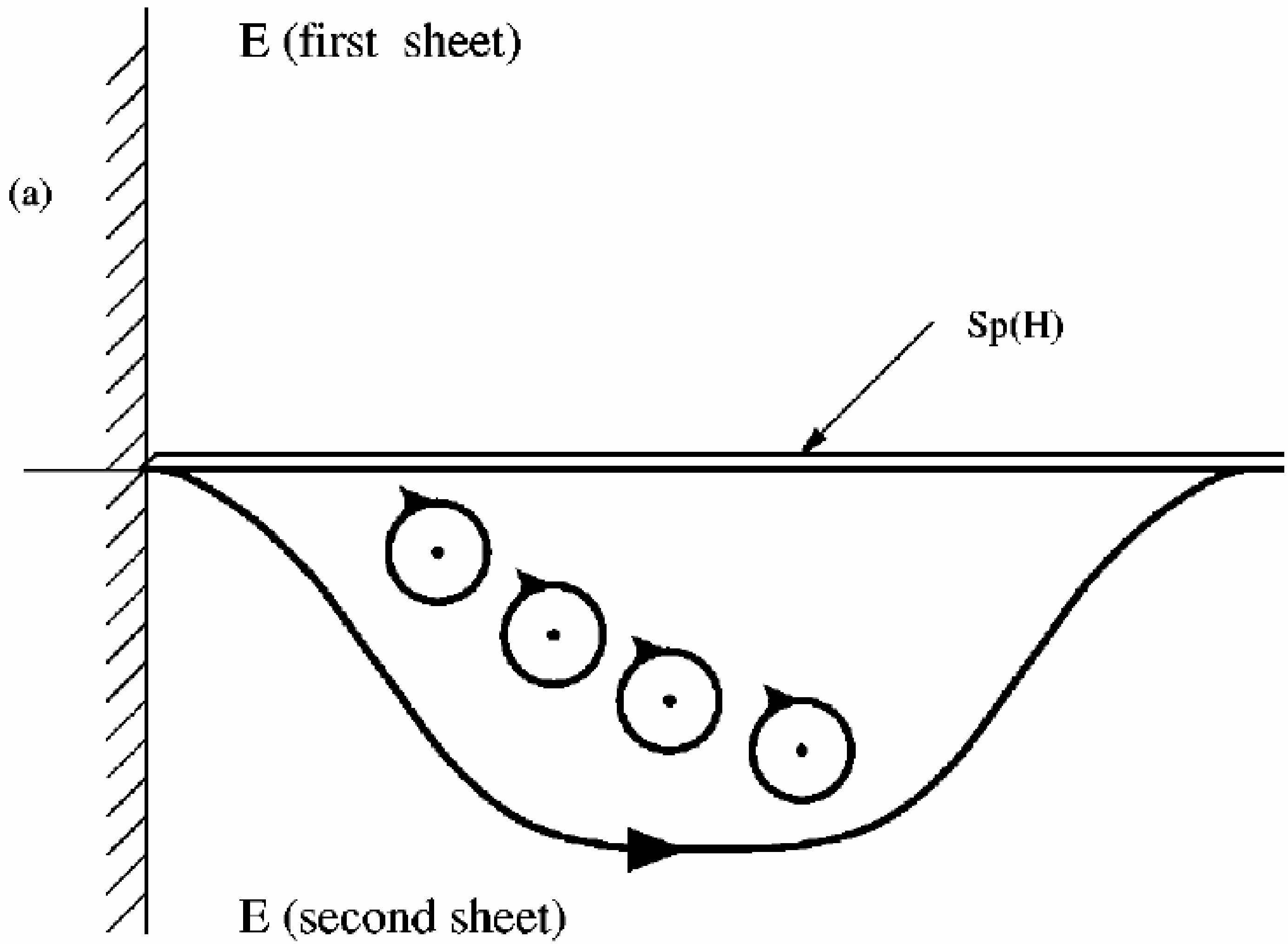
$$\varphi^{+} = \int_{0}^{-\infty} |E^{+}\rangle\langle {}^{+}E|\varphi^{+}\rangle dE$$

$$-2\pi i \sum_{n=0}^{\infty} r_n |z_{d,n}^{-}\rangle\langle {}^{+}z_{d,n}|\varphi^{+}\rangle.$$

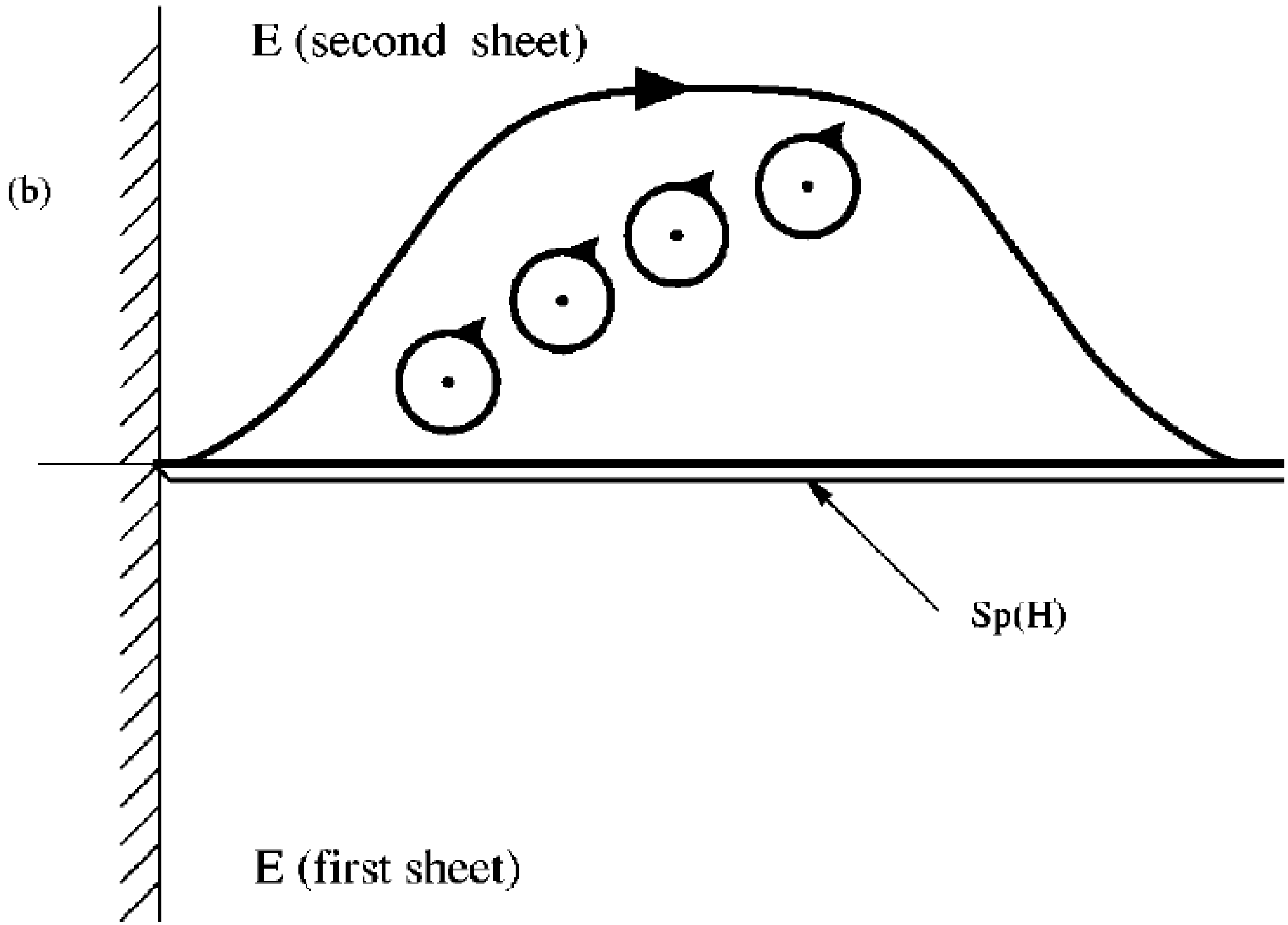
and a similar expression holds for the out-state

$$\psi^{-} = \int_{0}^{\infty} |E^{-}\rangle\langle E^{-}|\psi^{-}\rangle dE$$

$$+2\pi i \sum_{n=0}^{\infty} r_n^{*} |z_{g,n}^{*}{}^{+}\rangle\langle {}^{-}z_{g,n}^{*}|\psi^{-}\rangle$$



Contribution from decaying Gamow vectors



Contribution from growing Gamow vectors

Time asymmetry of the purely outgoing boundary condition:

for kd=Re(k)-iIm(k); with Re(k), Im(k)>0 (fourth quadrant) we write

$$\chi_{\text{incoming}}^{\text{decaying}}(r,t) = \mathcal{F}_2 e^{-ik_d r} e^{-iz_d t/\hbar}$$
$$= (\mathcal{F}_2 e^{-\mathrm{Im}(k) r - \Gamma_R t/(2\hbar)}) e^{-i\,\mathrm{Re}(k) r - iE_R t/\hbar}$$
$$r > b,$$

and

$$\chi_{\text{outgoing}}^{\text{decaying}}(r,t) = \mathcal{F}_1 e^{-ik_d r} e^{-iz_d t/\hbar}$$
$$= (\mathcal{F}_1 e^{\mathrm{Im}(k) r - \Gamma_R t/(2\hbar)}) e^{-i\,\mathrm{Re}(k) r - iE_R t/\hbar},$$
$$r > b,$$

In the same fashion, for the growing vectors we have for kg=-Re(k)-iIm(k); (with Re(k), Im(k)>0, third quadrant)

$$\chi^{\text{growing}}_{\text{incoming}}(r,t) = \mathcal{F}_1 e^{+ik_g r} e^{-iz_g t/\hbar}$$
$$= (\mathcal{F}_1 e^{\text{Im}(k)r + \Gamma_R t/(2\hbar)}) e^{-i\,\text{Re}(k)r - iE_R t/\hbar}$$
$$r > b,$$

and

$$\chi^{\text{growing}}_{\text{outgoing}}(r,t) = \mathcal{F}_2 e^{-ik_g r} e^{-iz_g t/\hbar}$$
$$= (\mathcal{F}_2 e^{-\text{Im}(k)r + \Gamma_R t/(2\hbar)}) e^{i\,\text{Re}(k)r - iE_R t/\hbar}$$
$$r > b,$$

Leading to the probability densities

$$\begin{aligned}
\rho^{\text{decaying}}_{\text{incoming}}(r,t) &= |\chi^{\text{decaying}}_{\text{incoming}}(r,t)|^2 \\
&= |\mathcal{F}_2|^2 e^{-2\ \mathrm{Im}(k) r - \Gamma_R t/\hbar} \\
&= |\mathcal{F}_2|^2 e^{-\Gamma_R/\hbar (t + r/v)}, \quad r > b
\end{aligned}$$

$$\begin{aligned}
\rho^{\text{decaying}}_{\text{outgoing}}(r,t) &= |\chi^{\text{decaying}}_{\text{outgoing}}(r,t)|^2 \\
&= |\mathcal{F}_1|^2 e^{2\ \mathrm{Im}(k) r - \Gamma_R t/\hbar} \\
&= |\mathcal{F}_1|^2 e^{-\Gamma_R/\hbar (t - r/v)}, \quad r > b
\end{aligned}$$

and

$$\begin{aligned}\rho_{\text{incoming}}^{\text{growing}}(r,t) &= |\chi_{\text{incoming}}^{\text{growing}}(r,t)|^2 \\ &= |\mathcal{F}_1|^2 e^{2\ \mathrm{Im}(k) r + \Gamma_R t/\hbar} \\ &= |\mathcal{F}_1|^2 e^{\Gamma_R/\hbar(t + r/v)}, \quad r > b\end{aligned}$$

$$\begin{aligned}\rho_{\text{outgoing}}^{\text{growing}}(r,t) &= |\chi_{\text{outgoing}}^{\text{growing}}(r,t)|^2 \\ &= |\mathcal{F}_2|^2 e^{-2\ \mathrm{Im}(k) r + \Gamma_R t/\hbar} \\ &= |\mathcal{F}_2|^2 e^{\Gamma_R/\hbar(t - r/v)}, \quad r > b\end{aligned}$$

Rigged Hilbert Space treatment of continuum spectrum

- Mathematical concepts
- Observables

Reference: O. Civitarese and M. Gadella Phys. Rep. 396 (2004) 41, and references quoted therein.

The Hilbert space of scattering states

We can write the Hilbert space of a self-adjoint (central) Hamiltonian as the direct orthogonal sum of the discrete (d) and continuous (c) subspaces

$$\mathscr{H} = \mathscr{H}_{\mathrm{d}} \oplus \mathscr{H}_{\mathrm{c}}$$

And, moreover, we can decompose the continuous subspace into two mutually orthogonal parts (absolutely continuous (ac) and singular continuous (sc) parts)

$$\mathscr{H}_{\mathrm{c}} = \mathscr{H}_{\mathrm{ac}} \oplus \mathscr{H}_{\mathrm{sc}}$$

Scattering states are not-bound regular states and they belong to to the ac part of the spectrum, other states are contained in sc (fractal section of the spectrum of H), thus

$$\mathscr{H} = \mathscr{H}_{\mathrm{d}} \oplus \mathscr{H}_{\mathrm{ac}} \oplus \mathscr{H}_{\mathrm{sc}}$$

The Moller operator

We assume that any scattering state is asymptotically free in the past. For any scattering state , φ it exists a free state ψ such

$$\lim_{t\to-\infty} \|e^{-itH}\phi - e^{-itH_0}\psi\| = 0$$

As the limit in a Hilbert space is taken with respect to its norm, this is equivalent to say that

$$\lim_{t\to-\infty} \{e^{-itH}\phi - e^{-itH_0}\psi\} = 0\ .$$

Since the evolution operator is unitary, we have:

$$\lim_{t\to-\infty} \|\phi - \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}\psi\| = 0$$

Then, we can de<ne an operator which relates each scattering state with its corresponding asymptotically free state

$$\phi := \boldsymbol{\Omega}_{\mathrm{OUT}}\psi = \lim_{t\to-\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}\psi$$

Analogously, we also assume that any scattering state is asymptotically free in the future. This means that for any scattering state, there exists a free state such that

$$\lim_{t\to\infty} \{\mathrm{e}^{-\mathrm{i}tH}\phi - \mathrm{e}^{-\mathrm{i}tH_0}\varphi\} = 0$$

then, there exists an operator

$$\phi = \mathbf{\Omega}_{\mathrm{IN}}\varphi = \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}\varphi$$

These operators are the Moller wave operators

$$\boldsymbol{\Omega}_{\mathrm{IN}}^{-1} = \boldsymbol{\Omega}_{\mathrm{IN}}^{\dagger}, \qquad \boldsymbol{\Omega}_{\mathrm{OUT}}^{-1} = \boldsymbol{\Omega}_{\mathrm{OUT}}^{\dagger}$$

with the properties

$$\boldsymbol{\Omega}\psi(0) = \varphi(0)$$

$$\boldsymbol{\Omega}\psi(\tau) = \varphi(\tau)$$

Proof:

$$e^{itH}e^{-itH_0}\psi(\tau) = e^{itH}e^{-i(t+\tau)H_0}\psi(0) = e^{-i\tau H}e^{i(t+\tau)H}e^{-i(t+\tau)H_0}\psi(0)$$

$$\boldsymbol{\Omega}\psi(\tau) = e^{-i\tau H}\varphi(0) = \varphi(\tau)$$

$$\boldsymbol{\Omega}e^{-i\tau H_0}\psi(0) = e^{-i\tau H}\varphi(0) = e^{-i\tau H}\boldsymbol{\Omega}\psi(0)$$

$$\boldsymbol{\Omega}e^{-i\tau H_0} = e^{-i\tau H}\boldsymbol{\Omega}$$

$$\boldsymbol{\Omega}H_0 = H\boldsymbol{\Omega}$$

The evolution of the state, from the initial prepared state to the scattered state is given by

$$\varphi(t) = \boldsymbol{\Omega}_{\mathrm{OUT}}^{-1} \boldsymbol{\Omega}_{\mathrm{IN}} \psi(t)$$

Then it seems natural to define the S matrix in terms of Moller operators

$$S := \boldsymbol{\Omega}_{\mathrm{OUT}}^{-1} \boldsymbol{\Omega}_{\mathrm{IN}}$$

so that

$$S \mathrm{e}^{-\mathrm{i}tH_0} \psi(0) = \mathrm{e}^{-\mathrm{i}tH_0} \varphi(0) = \mathrm{e}^{-\mathrm{i}tH_0} S \psi(0)$$

With this choice

$$Se^{-\mathrm{i}tH_0} = e^{-\mathrm{i}tH_0}S$$

Which is equivalent, in infinitesimal form, to

$$SH_0 = H_0S \quad \Leftrightarrow \quad [S, H_0] = 0$$

Analytic continuation of S(E)->S(p=+$\sqrt{2mE}$)
(complex p-plane or k-plane)

(i) Single poles in the positive imaginary axis of S(z) that correspond to the bound states of H.
(ii) Single poles in the negative imaginary axis that correspond to virtual states.
(iii) Pairs of poles, in principle of any order, in the lower half-plane. Each of the poles ofa pair
has the same negative imaginary part and the same real part with opposite sign.
Thus, if p is one of these two poles the other is $-p^*$ (complex conjugation). These poles are
called resonance poles and in general there is an infinite number ofthem

Rigged Hilbert spaces (RHS)

We shall define resonance states as eigenvectors of H with complex eigenvalues, located at the resonance poles.

As self-adjoint operators in Hilbert spaces do not have complex eigenvalues, resonant states cannot be vectors on a Hilbert space.

They belong to certain extensions of Hilbert spaces which are the rigged Hilbert spaces (RHS).

We start with a definition of RHS:

$$\boldsymbol{\Phi} \subset \mathscr{H} \subset \boldsymbol{\Phi}^{\times}$$

A triplet of spaces is a rigged Hilbert space (RHS) if:

$\mathscr{H}$ The intermediate space H is an infinite-dimensional Hilbert space.

$\boldsymbol{\Phi}$ Is a topological vector space, which is dense in H (we call this space left-sail space of RHS)

$\boldsymbol{\Phi}^{\times}$ Is the anti-dual space of $\boldsymbol{\Phi}$

(we call this space : right-sail space of RHS)

RHS are useful, among other applications, for:

1. Giving a rigorous meaning to the Dirac formulation of quantum mechanics .

   In this case,it is customary to demand that the vector space on the left be nuclear(e.g.; obey the Nuclear Spectral Theorem, according to which every observable, or set of commuting observables, has a complete set of generalized eigenvectors whose corresponding eigenvalues exhaust the whole spectrum of the observable. This is in agreement with the Dirac requirement

2. Giving a proper mathematical meaning to the Gamow vectors, i.e., vector states which represent resonances
3. Extending quantum mechanics to accommodate the irreversible character of certain quantum processes such as decay processes.
4. Dealing with physical problems requiring the use of distributions. In fact, distributions are well known to be objects in the dual of a nuclear locally convex space

Rigged Hilbert spaces have the following properties:

Property 1. Let A be an operator on H. If we define the domain (D) of its adjoint, then the following conditions are fulfilled:
(i) The domain contains the left-sail space of the RHS.
(ii) For each function belonging to the left-sail space the action of the adjoint operator on it also belongs to the left-sail space.
(iii) The adjoint operator is continuous on the left-sail space . Then

$$\langle A^{\dagger}\varphi|F\rangle = \langle\varphi|AF\rangle, \quad \forall\varphi\in\boldsymbol{\Phi},\ \forall F\in\boldsymbol{\Phi}^{\times}$$

Property 2. Let A be an operator with the properties described in Property 1. A complex number λ is a generalized eigenvalue of A if for any function φ belonging to the left-sail space and for some non-zero F belonging to the right-sail space we have that

$$\langle A^{\dagger}\varphi|F\rangle = \lambda\langle\varphi|F\rangle$$

$$\langle\varphi|AF\rangle = \lambda\langle\varphi|F\rangle$$

then

$$AF = \lambda F$$

Property 3. A result due to Gelfand and Maurin states the following: Let A be a self-adjoint operator on H with continuous spectrum $\sigma(A)$. Although it is not necessary, we may assume that the spectrum is purely continuous. Then, there exists a RHS, such that

(i) A can be extended by the duality formula to the anti-dual right-sail space.

(ii) There exists a measure dμ. on $\sigma(A)$, which can be chosen to be the Lebesgue measure if the spectrum is absolutely continuous, such that for almost all $\lambda \in \sigma(A)$ with respect to dμ, there exists a nonzero functional $F(\lambda)$ of the right-sail space such that

$$AF_\lambda = \lambda F_\lambda$$

This means that the points in the continuous spectrum of A are eigenvectors of the extension of A into the right-sail space. However, these eigenvectors do not belong to the Hilbert space.

From the previous properties of RHS we have that for all φ and ψ (of the left-sail space)

$$\langle \varphi | A\phi \rangle = \int_{\sigma(A)} \lambda F_\lambda(\varphi)[F_\lambda(\phi)]^* \,\mathrm{d}\mu$$

$$\langle \varphi | \phi \rangle = \int_{\sigma(A)} F_\lambda(\varphi)[F_\lambda(\phi)]^* \,\mathrm{d}\mu \ .$$

$$\langle \varphi | f(A)\phi \rangle = \int_{\sigma(A)} f(\lambda) F_\lambda(\varphi)[F_\lambda(\phi)]^* \,\mathrm{d}\mu$$

Then, in Dirac notation:

$$\langle \varphi | f(A)\phi \rangle = \int_{\sigma(A)} f(\lambda) \langle \varphi | \lambda \rangle \langle \lambda | \phi \rangle \, \mathrm{d}\lambda$$

$$\langle \lambda | \phi \rangle = \langle \phi | \lambda \rangle^*$$

$$f(A) = \int_{\sigma(A)} f(\lambda) |\lambda\rangle \langle \lambda| \, \mathrm{d}\lambda$$

$$H|E_+\rangle = H\boldsymbol{\Omega}_{\mathrm{OUT}}|E\rangle = \boldsymbol{\Omega}_{\mathrm{OUT}}H_0|E\rangle = E\boldsymbol{\Omega}_{\mathrm{OUT}}|E\rangle = E|E_+\rangle,$$
$$H|E_-\rangle = H\boldsymbol{\Omega}_{\mathrm{IN}}|E\rangle = \boldsymbol{\Omega}_{\mathrm{IN}}H_0|E\rangle = E\boldsymbol{\Omega}_{\mathrm{IN}}|E\rangle = E|E_-\rangle \ .$$

$$\langle\phi_+|E_+\rangle = \langle\boldsymbol{\Omega}_{\mathrm{OUT}}\phi^{\mathrm{out}}|\boldsymbol{\Omega}_{\mathrm{OUT}}|E\rangle = \langle\phi^{\mathrm{out}}|E\rangle = [\phi^{\mathrm{out}}(E)]^*$$

$$\langle\psi_-|E_-\rangle = \langle\boldsymbol{\Omega}_{\mathrm{IN}}\psi^{\mathrm{in}}|\boldsymbol{\Omega}_{\mathrm{IN}}|E\rangle = \langle\psi^{\mathrm{in}}|E\rangle = [\psi^{\mathrm{in}}(E)]^* \ .$$

$$\langle \phi_+ | \psi_- \rangle = \langle \phi^{\text{out}} | S \psi^{\text{in}} \rangle = \int_0^\infty \langle \phi^{\text{out}} | E \rangle S(E) \langle E | \psi^{\text{in}} \rangle \, \mathrm{d}E$$

$$= \int_0^\infty [\phi^{\text{out}}(E)]^* S(E) \psi^{\text{in}}(E) \, \mathrm{d}E \ .$$

$$\begin{aligned}\int_0^R [\phi^{\text{out}}(E)]^* S(E) \psi^{\text{in}}(E) \, \mathrm{d}E = & - \int_{-R}^0 [\phi^{\text{out}}(E)]^* S(E) \psi^{\text{in}}(E) \, \mathrm{d}E \\ & + \int_C [\phi^{\text{out}}(z^*)]^* S(z) \psi^{\text{in}}(z) \, \mathrm{d}z \\ & - 2\pi \mathrm{i} \sum \text{Residues}\{[\phi^{\text{out}}(z)]^* S(z) \psi^{\text{in}}(z)\} \ ,\end{aligned}$$

where
(i) The integral over the negative axis refers to the negative axis in the second sheet of the Riemann surface.
(ii) C is the semicircle, in the lower half-plane of the second sheet, centered at the origin with radius R, which does not contain any pole of S(E).

(iii) The sum ofthe residues extends over all poles of S(E) in the region limited by the contour [-R,R]

The mathematical concepts presented in this section are related to the structure of RHS and the construction of RHS for relevant examples. The transformations $\theta_\pm$ needed to define the basic spaces Φ, or space of test vectors, and $\Phi^\times$, its dual, for certain cases of physical interest, have been presented. A brief list of the relevant objects is the following:

$$\mathscr{H} = \bigoplus_n \mathscr{H}_n \quad \text{(Hilbert space)} ,$$

$$\boldsymbol{\Omega}_{\mathrm{OUT}}, \quad \boldsymbol{\Omega}_{\mathrm{IN}} \quad \text{(Møller wave operators)} ,$$

$$S = \boldsymbol{\Omega}_{\mathrm{IN}}\boldsymbol{\Omega}^\dagger_{\mathrm{OUT}} \quad (S\text{-operator}) ,$$

$$\boldsymbol{\Phi} \subset \mathscr{H} \subset \boldsymbol{\Phi}^\times \quad \text{(rigged Hilbert spaces)} ,$$

$$\mathscr{H}^2_\pm \quad \text{(Hardy spaces)} ,$$

$$S \cap \mathscr{H}^2_\pm \quad \text{(Hardy–Schwartz Functions)} ,$$

$$\theta_\pm, \theta^\times_\pm \quad \text{(Mappings)} ,$$

$$S \cap \mathscr{H}^2_\pm|_{\mathbb{R}^+} \quad (\text{Restrictions to } \mathbb{R}^+ \text{ of functions in } S \cap \mathscr{H}^2_\pm) .$$

$$
\begin{tikzcd}
\boldsymbol{\Phi}_- \arrow[d, "\boldsymbol{\Omega}_{\mathrm{IN}}^{-1}"'] & \subset & \mathscr{H}_{\mathrm{ac}}(H) \arrow[d, "\boldsymbol{\Omega}_{\mathrm{IN}}^{-1}"] & \subset & \boldsymbol{\Phi}_-^{\times} \arrow[d, "\boldsymbol{\Omega}_{\mathrm{cIN}}^{-1}"] \\
\boldsymbol{\Psi}_- \arrow[d, "U"'] & \subset & \mathscr{H}_{\mathrm{ac}}(H_0) \arrow[d, "U"] & \subset & \boldsymbol{\Psi}_-^{\times} \arrow[d, "U"] \\
S \cap \mathscr{H}_-^2|_{\mathbb{R}^+} \arrow[d, "\theta_-^{-1}"'] & \subset & L^2(\mathbb{R}^+) & \subset & (S \cap \mathscr{H}_-^2|_{\mathbb{R}^+})^{\times} \arrow[d, "(\theta_-^{-1})^{\times}"] \\
S \cap \mathscr{H}_-^2 & \subset & L^2(\mathbb{R}) & \subset & (S \cap \mathscr{H}_-^2)^{\times}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boldsymbol{\Phi}_+ \arrow[r, phantom, "\subset"] \arrow[d, "\boldsymbol{\Omega}_{\mathrm{OUT}}^{-1}"'] & \mathscr{H}_{\mathrm{ac}}(H) \arrow[r, phantom, "\subset"] \arrow[d, "\boldsymbol{\Omega}_{\mathrm{OUT}}^{-1}"] & \boldsymbol{\Phi}_+^{\times} \arrow[d, "\boldsymbol{\Omega}_{\mathrm{OUT}}^{-1}"] \\
\boldsymbol{\Psi}_+ \arrow[r, phantom, "\subset"] \arrow[d, "U"'] & \mathscr{H}_{\mathrm{ac}}(H_0) \arrow[r, phantom, "\subset"] \arrow[d, "U"] & \boldsymbol{\Psi}_+^{\times} \arrow[d, "U"] \\
S \cap \mathscr{H}_+^2|_{\mathbb{R}^+} \arrow[r, phantom, "\subset"] \arrow[d, "\theta_+^{-1}"'] & L^2(\mathbb{R}^+) \arrow[r, phantom, "\subset"] & (S \cap \mathscr{H}_+^2|_{\mathbb{R}^+})^{\times} \arrow[d, "(\theta_+^{-1})^{\times}"] \\
S \cap \mathscr{H}_+^2 \arrow[r, phantom, "\subset"] & L^2(\mathbb{R}) \arrow[r, phantom, "\subset"] & (S \cap \mathscr{H}_+^2)^{\times}
\end{tikzcd}
$$

Normalization and mean values

- The problem: how to evaluate mean value of operators, like the Hamiltonian, on Gamow vectors?. Since the inner product (scalar product) of Gamow vectors is not defined, we have several possibilities, namely:
- The mean values are zero (N. Nakanishi, Prog.Theor.Phys. 19 (1958) 607)
- The mean values are complex (M. Gadella, Int. Journal. Theor.Phys. 36 (1997) 2271.
- The mean values are real (on the RHS, A. Bohm and M. Gadella, Dirac kets, Gamow vectors and Gelfant triplets; Springer-Verlag (1989))
- The mean value are the result of interferences between capturing and decaying vectors (T. Berggren; Phys. Lett. B 373 (1996) 1.)

The mean values are zero

If we define the action of the Hamiltonian on the Gamow vector as:

$$H|f_0\rangle = z_R|f_0\rangle \qquad \langle f_0|H = z_R^*\langle f_0|$$

$$\langle f_0|H|f_0\rangle = z_R\langle f_0|f_0\rangle = z_R^*\langle f_0|f_0\rangle$$

$$(z_R - z_R^*)\langle f_0|f_0\rangle = 0 \Rightarrow \langle f_0|f_0\rangle = 0 \text{ and, therefore, } \langle f_0|H|f_0\rangle = 0.$$

Comment: the object $\langle f_0|f_0\rangle$

Is not defined, since the vectors do not belong to the Hilbert space of H.

•The mean values are complex

In specific models, like Friedrichs's model, the bracket $\langle \tilde{f}_0 | f_0 \rangle$ is well defined

$$\Pi = |f_0\rangle\langle \tilde{\tilde{f}}_0|$$

$$\overline{\mathrm{Tr}}\{H\Pi\} = \langle \tilde{f}_0 | H | f_0 \rangle$$

$$\langle \tilde{f}_0 | H | f_0 \rangle = z_R \langle \tilde{f}_0 | f_0 \rangle = z_R$$

Comment: physically un-acceptable, since we cannot determine simultaneously the real and imaginary part of the expectation value

The mean values are real

For a mapping of analytic functions (like projections on the positive real semi-axis) the norms are finite,

$$\|\theta_{\pm}\varphi_{\pm}\|_{L^2(R^+)} = \int_0^{\infty} |\varphi_{\pm}(E)|^2 \, dE < \int_{-\infty}^{\infty} |\varphi_{\pm}(E)|^2 \, dE = \|\varphi_{\pm}\|_{L^2(R)} < \infty .$$

$$\langle \varphi_{+} | f_0 \rangle = \phi_{+}^{\#}(z_R^*) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi_{+}^{\#}(E)}{E - z_R^*} \, dE$$

and

$$\langle \varphi_{-} | \tilde{f}_0 \rangle = \phi_{-}^{\#}(z_R) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi_{-}^{\#}(E)}{E - z_R} \, dE .$$

$$\psi^D = \alpha \frac{1}{E - z_R}, \qquad \psi^G = \alpha \frac{1}{E - z_R^*},$$

then

$$\|\psi^D\|^2 = \alpha^2 \int_{-\infty}^{\infty} \frac{dE}{(E - z_R)^2 + (\Gamma/2)^2} = \alpha^2 \pi .$$

Therefore, $\|\psi^D\| = \|\psi^G\| = 1$ if $\alpha = 1/\sqrt{\pi}$.

$$\hat{E} \frac{1}{E - z_R} = \frac{E}{E - z_R}$$

$$\langle \psi^D | \hat{E} | \psi^D \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{E - z_R^*} \frac{E}{E - z_R} dE = \frac{2}{\pi \Gamma} \int_{-\infty}^{\infty} \frac{E\, dE}{\left(\dfrac{E - E_R}{\Gamma/2} \right)^2 + 1} .$$

Changing variables to

$$x = \frac{E - E_R}{\Gamma/2}$$

the integral transforms like

$$\frac{E_R}{\pi}\int_{-\infty}^{\infty}\frac{dx}{x^2+1} + \frac{\Gamma}{2\pi}\int_{-\infty}^{\infty}\frac{x\,dx}{x^2+1}\,.$$

Thus we find

$$\langle \psi^D | \hat{E} | \psi^D \rangle = E_R$$

and

$$\langle \psi^G | \hat{E} | \psi^G \rangle = E_R\,.$$

The mean value are the result of interferences between capturing and decaying vectors

$$\psi(E) = \psi_u(E + i0) - \psi_l(E - i0)\,,$$

$$\psi(z) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi(E)}{E - z}\,dE\,,$$

$$\psi(z_R) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi(E)}{E - z_R}\,dE\,, \qquad \psi(z_R^*) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi(E)}{E - z_R^*}\,dE\,.$$

$$|f_0\rangle = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{|E\rangle\, dE}{E - z_R}\,, \qquad |\tilde{f}_0\rangle = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{|E\rangle\, dE}{E - z_R^*}\,.$$

In momentum space (Berggren) one has

$$|f_0\rangle = i\sqrt{\frac{2\Gamma}{\pi}}\int_0^\infty \sqrt{\frac{k}{m}}\ \frac{|k,\hat{k},l\rangle}{E(\boldsymbol{k}) - z_R}\,dk$$

$$|\tilde{f}_0\rangle = -i\sqrt{\frac{2\Gamma}{\pi}}\int_0^\infty \sqrt{\frac{k}{m}}\ \frac{|k,\hat{k},l\rangle}{E(\boldsymbol{k}) - z_R^*}\,dk$$

$$\langle f_0|A|f_0\rangle = \frac{2\Gamma}{\pi}\sum_{l,l'}\int_0^\infty dk\int_0^\infty dk'\ \frac{\sqrt{kk'}}{m}\ \frac{\langle k',\hat{k}',l'|A|k,\hat{k},l\rangle}{(E(\boldsymbol{k}') - z_R)(E(\boldsymbol{k}) - z_R^*)}\ .$$

$$\langle f_0|A|f_0\rangle = \mathrm{Real}\left\{\langle \tilde{f}_0|A|f_0\rangle\right\} + o(\Gamma^2)$$

This is Berggren’s main result.

To better understand the previous results we shall start from T. Berggren' work , and compare it with our own work

Delta-function normalization in momentum space, for a continuum wave function

$E = (\hbar k)^2/2\mu$ by

$$\Phi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r}) = \sqrt{\frac{2}{\pi}} \frac{k^{l+1}}{f_l(-k)r} \varphi_l(k,r) \sum_m Y_{lm}(\hat{r}) Y_{lm}^*(\hat{k}).$$

$$f_l^*(k^*) = f_l(-k), \quad \varphi_l(k^*,r) = \varphi_l^*(k,r),$$

$$\varphi_l(k,r) = \varphi_l(-k,r).$$

$$\varphi_l(k,r) = \tfrac{1}{2} i k^{-l-1} \times \{ f_l(-k) f_l(k,r) - (-)^l f_l(k) f(-k,r) \}$$

For a given operator

$$A(\boldsymbol{r},\boldsymbol{r}') = \sum_l \frac{A_l(r,r')}{rr'} \sum_m Y_{lm}(\hat{r}) Y^*_{lm}(\hat{r}')$$

$$\langle \Phi_{k^*} | A | \Phi_k \rangle = \frac{2}{\pi} \sum_l \int\limits_{r=0}^{\infty} \int\limits_{r'=0}^{\infty} \varphi_l^*(k^*, r) A_l(r, r') \varphi_l(k, r') \mathrm{d}r \mathrm{d}r' \times \frac{k^{2l+2}}{f_l(k) f_l(-k)} \sum_m Y_{lm}(\hat{k}) Y_{lm}^*(\hat{k}).$$

$$k_n = \kappa_n - i\gamma_n.$$

$$f_l(k_n)\left[\frac{\mathrm{d}f_l(k)}{\mathrm{d}k}\right]_{k=-k_n} = 4ik_n^{2l+2}\int_0^\infty \mathrm{d}r\,\varphi_l(k_n, r)^2$$

$$\equiv 4ik_n^{2l+2}N^2.$$

$$\begin{aligned} D_{ln}(k) &\equiv f_l(k)\,f_l(-k) = f_l^*(-k^*)\,f_l(-k) \\ &\approx (k-k_n^*)(k-k_n)|\dot{f}_l(-k_n)|^2 \\ &= [(k-\kappa_n)^2+\gamma_n^2]|\dot{f}_l(-k_n)|^2. \end{aligned}$$

$$\frac{1}{D_{ln}} \approx \frac{4\gamma_n^2}{[(k-\kappa_n)^2+\gamma_n^2]\,|f_l(k_n)|^2}.$$

$$\dot{f}_l(-k_n) \approx \frac{f_l(-k_n^*) - f_l(-k_n)}{-k_n^* + k_n} = \frac{f_l^*(k_n)}{-2i\gamma_n}$$

If we replace in the expectation value

$$a_l(k) \equiv k^{2l+2} \int_{r=0}^{\infty} \int_{r'=0}^{\infty} \varphi_l(k,r) A_l(r,r') \varphi_l^*(k^*,r') \mathrm{d}r \mathrm{d}r'$$

then

$$\begin{aligned} a_l(k) &= a_l(k_n) + (k - k_n)\dot{a}_l(k_n) + \ldots \\ &= a_l(\kappa_n) - i\gamma_n \dot{a}_l(\kappa_n) \\ &+ (k - \kappa_n + i\gamma_n)[\dot{a}_l(\kappa_n) - i\gamma_n \ddot{a}_l(\kappa_n)] + \ldots . \end{aligned}$$

to first order in $(k - k_n) \equiv (k - \kappa_n + i\gamma_n)$

$$\mathrm{Re}\{a_l(k_n)\} \approx a_l(\kappa_n), \quad \mathrm{Im}\{a_l(k_n)\} \approx -\gamma_n \dot{a}_l(\kappa_n)$$

$$\begin{aligned} a_l(k_n) &= k_n^{2l+2} N^2 \langle \tilde{u}_{nl} | A_l | u_{nl} \rangle \\ &= \frac{1}{4i} f_l(k_n) \dot{f}_l(-k_n) \langle \tilde{u}_{nl} | A_l | u_{nl} \rangle \\ &\approx \frac{|f_l(k_n)|^2}{8\gamma_n} \langle \tilde{u}_{nl} | A_l | u_{nl} \rangle. \end{aligned}$$

Finally,

$$\langle \Phi_{k^*} | A | \Phi_k \rangle \approx \frac{2}{\pi} \frac{4\gamma_n^2 [a_l(\kappa_n) + (k - \kappa_n)\dot{a}_l(\kappa_n)]}{[(k-\kappa_n)^2 + \gamma_n^2] |f_l(k_n)|^2}$$

$$\times \sum_m Y_{lm}(\hat{k}) Y_{lm}^*(\hat{k}) = \frac{\gamma_n}{\pi}$$

$$\times \frac{\mathrm{Re}\langle \tilde{u}_{nl} | A_l | u_{nl} \rangle - [(k-\kappa_n)/\gamma_n] \mathrm{Im}\langle \tilde{u}_{nl} | A_l | u_{nl} \rangle}{(k-\kappa_n)^2 + \gamma_n^2}$$

$$\times \sum_m Y_{lm}(\hat{k}) Y_{lm}^*(\hat{k})$$

From where it reads

$$\langle A \rangle_l = \frac{1}{\pi} \sum_n \mathrm{Re}\langle \tilde{u}_{nl} | A_l | u_{nl} \rangle \int_{s_{\min}}^{s_{\max}} \frac{\mathrm{d}s_n}{1 + s_n^2}$$

$$- \sum_n \frac{1}{\pi} \mathrm{Im}\langle \tilde{u}_{nl} | A_l | u_{nl} \rangle \int_{s_{\min}}^{s_{\max}} \frac{s_n \mathrm{d}s_n}{1 + s_n^2}.$$

Following Berggren's interpretation

$$\langle A \rangle = \mathrm{Re}\langle \tilde{\Phi}_n | A | \Phi_n \rangle.$$

$$\begin{aligned}(\Delta A)^2 &= \mathrm{Re}\langle \tilde{\Phi}_n | (A - \langle A \rangle)^2 | \Phi_n \rangle \\ &= \mathrm{Re}\langle \tilde{\Phi}_n | A^2 - 2A\langle A \rangle + \langle A \rangle^2 | \Phi_n \rangle.\end{aligned}$$

$$\begin{aligned}\langle \tilde{\Phi}_n | A^2 | \Phi_n \rangle &= \langle \tilde{\Phi}_n | A | \Phi_n \rangle \langle \tilde{\Phi}_n | A | \Phi_n \rangle \\ &\equiv (A - \mathrm{i}\,\mathrm{Im}\langle \tilde{\Phi}_n | A | \Phi_n \rangle)^2.\end{aligned}$$

$$(\Delta A)^2 = -(\mathrm{Im}\langle \tilde{\Phi}_n | A | \Phi_n \rangle)^2.$$

In our approach

$$\hat{\psi}_c(z) = \frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} \frac{1}{t - z} \hat{\psi}(t)\, dt\,,$$

where $\hat{\psi}(t) = \hat{\psi}_c(t + i0) - \hat{\psi}_c(t - i0)$.

$$\left(\hat{\psi}(E_G)\right)^* = \frac{1}{2\pi i} \int\limits_{E_0}^{E_1} \frac{1}{{E_G}^* - E} \left(\hat{\psi}(E)\right)^* dE$$

with $E_G = E_D + i\Gamma$, $\Gamma > 0$.

and

$$(\hat{\psi}(E_G))^* = \frac{1}{2\pi i}\int_{E_0}^{E_1} \frac{1}{E_G{}^* - E} <\psi|E> dE\,.$$

$$|E_G{}^*> = \frac{1}{2\pi i}\int_{E_0}^{E_1} \frac{1}{E_G{}^* - E}\,|E> dE\,.$$

$$<\psi|H|E_G{}^*> = \frac{1}{2\pi i}\int_{E_0}^{E_1} \frac{1}{E_G{}^* - E} E(\psi(E))^*\, dE$$

$$= E_G{}^* <\psi|E_G{}^*> .$$

$$<E_G|E_G^*> = \frac{1}{4\pi^2\Gamma} \times \left[\arctan\left(\frac{E_1 - E_D}{\Gamma}\right) - \arctan\left(\frac{E_0 - E_D}{\Gamma}\right)\right]$$

$$<E_G|H|E_G^*> = E_D + \frac{\Gamma}{2}\frac{\ln\left[\frac{(E_1-E_D)^2+\Gamma^2}{(E_0-E_D)^2+\Gamma^2}\right]}{\left[\arctan\left(\frac{E_1-E_D}{\Gamma}\right) - \arctan\left(\frac{E_0-E_D}{\Gamma}\right)\right]}$$

$$P(E) = |\langle E|E_G^*\rangle|^2 = \frac{\Gamma}{(E-E_D)^2+\Gamma^2} \times \frac{1}{\left[\arctan\left(\frac{E_1-E_D}{\Gamma}\right) - \arctan\left(\frac{E_0-E_D}{\Gamma}\right)\right]}.$$

As an example, let us explore the consequences of the procedure for an arbitrary self-adjoint operator

$$A = \int_{-\infty}^{+\infty} |\lambda > \lambda \, d\sigma_a(\lambda) < \lambda| \, ,$$

$$\sigma_a(\lambda) = \begin{cases} \sum_{n=-\infty}^{+\infty} \Theta(\lambda - \lambda_n) & \lambda < \lambda_\infty \\ \lambda & \lambda_\infty < \lambda < \lambda_a \end{cases}$$

then

$$<E_G|A|E_G{}^*> = \int_{-\infty}^{+\infty} <E_G|\lambda > \lambda \, d\sigma_a(\lambda) <\lambda|E_G{}^*> \, ,$$

This value is real, since:

$$<E_G|\lambda> = (<\lambda|E_G^*>)^*.$$

$$|E_G{}^*> = \frac{\sqrt{\Gamma}}{i\sqrt{\pi/2}} \int_0^{+\infty} \frac{|E(\boldsymbol{k}), \hat{k}, l>}{E_G{}^* - E} \, dE\,,$$

$$|E_G{}^*> = \frac{\sqrt{\Gamma}}{i\sqrt{\pi/2}} \int_0^{+\infty} \sqrt{\frac{k}{m}} \, \frac{|k, \hat{k}, l>}{E_G{}^* - E(\boldsymbol{k})} \, dk\,.$$

$$<E_G|A|E_G{}^*> = \frac{2\Gamma}{\pi} \sum_{l,l'} \int_0^{+\infty} dk \int_0^{+\infty} dk' \frac{\sqrt{kk'}}{m}$$

$$\times \frac{<k', \hat{k}', l'|A|k, \hat{k}, l>}{(E(\boldsymbol{k}') - E_G)(E(\boldsymbol{k}) - E_G{}^*)} .$$

In Berggren's formalism

$$<A> = \mathrm{Re} <E_G^*|A|E_G^*>,$$

$$<E_G^*|A|E_G^*> = \frac{2\Gamma}{\pi} \sum_{l,l'} \int_0^{+\infty} dk \int_0^{+\infty} dk' \frac{\sqrt{kk'}}{m}$$

$$\times \frac{< k', \hat{k}', l'|A|k, \hat{k}, l >}{(E(\boldsymbol{k}') - E_G^*)(E(\boldsymbol{k}) - E_G^*)} .$$

$$
\begin{aligned}
<A> = <E_G|A|E_G^*> &= \mathrm{Re}<E_G^*|A|E_G^*> \\
&- \frac{2i\Gamma^2}{\pi}\sum_{l,l'}\int_0^{+\infty} dk \int_0^{+\infty} dk' \frac{\sqrt{kk'}}{m}\left[E(\boldsymbol{k}) - E(\boldsymbol{k}')\right] \\
&\times \frac{< k', \hat{k}', l' |A| k, \hat{k}, l >}{|E(\boldsymbol{k}') - E_G|^2 |E(\boldsymbol{k}) - E_G|^2} \\
&+ \frac{4\Gamma^3}{\pi}\sum_{l,l'}\int_0^{+\infty} dk \int_0^{+\infty} dk' \frac{\sqrt{kk'}}{m} \\
&\times \frac{< k', \hat{k}', l' |A| k, \hat{k}, l >}{|E(\boldsymbol{k}') - E_G|^2 |E(\boldsymbol{k}) - E_G|^2} .
\end{aligned}
$$

$$
<E_G|A|E_G^*> \neq \mathrm{Re}<E_G^*|A|E_G^*>
$$

Summary

- The quantum mechanical expectation value of a hermitean operator, when the system is in a resonant state, may be derived from the corresponding values defined for the continuum states having energies close to the resonance energy. (Berggren's result)
- When resonant Gamow states are constructed in a rigged Hilbert space, starting from Dirac's formula, the expectation value of a self-adjoint operator acting on a Gamow state is real. (Our result)

Some applications of Gamow vectors

- Illustration of resonances in a simple quantum mechanical problem
- Zeno paradox in quantum theory
- Alpha decay (revisited)
- The Friedrichs Model
- Nuclear structure and nuclear reactions with Gamow vectors

Some illustrative references

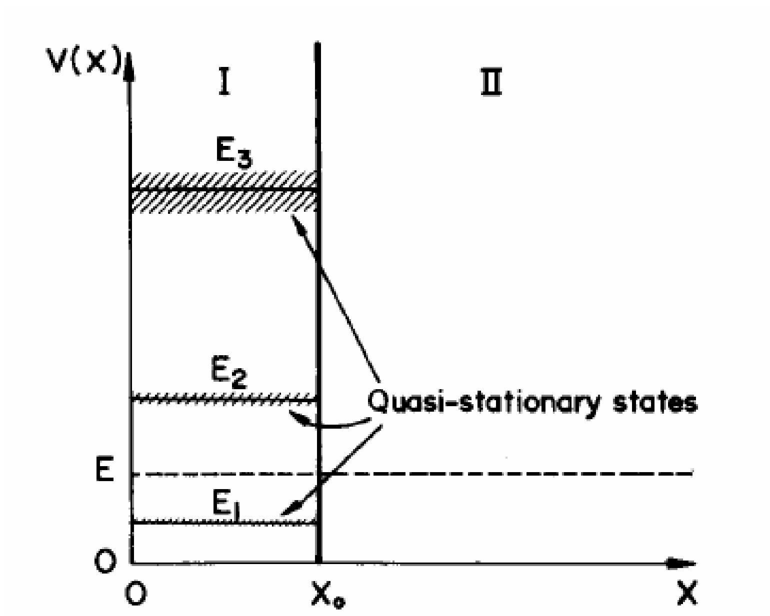
- H. Massmann ; Am.J.Phys 53 (1985) 679
- A. Peres; Am. J. Phys. 48 (1980) 931
- B. Holstein; Am. J. Phys. 64 (1996) 1061
- M. Baldo, L. S. Ferreira and L. Streit; Phys. Rev. C 36 (1987) 1743
- E. Hernandez and M. Mondragon; Phys. Rev. C 29 (1984) 722
- R. J. Liotta et al; Phys. Rev. Lett. 89 (2002)042501

and several other references from other authors and from myself… too lazy to compile them here, sorry…I will quoted them in the written version of the lectures….

A simple example (from H. Massmann; Am. J. Phys. 53 (1985) 679))

Let the potential be a one dimensional delta-barrier

$$V(x) = \begin{cases} \infty & x < 0 \\ W_0 \delta(x - x_0) & x \geqslant 0 \end{cases}.$$



$$\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}+W_0\delta(x-x_0)-E\right)\psi(x)=0$$

$$\psi_k^{(0)}(x)=\left(\frac{2}{\pi}\right)^{1/2}\sin(kx) \qquad \text{If } W_0=0$$

$$k=+(2\mu E)^{1/2}$$

$$\int_0^\infty \psi_k^{(0)*}(x)\psi_{k'}^{(0)}(x)dx=\delta(k-k')$$

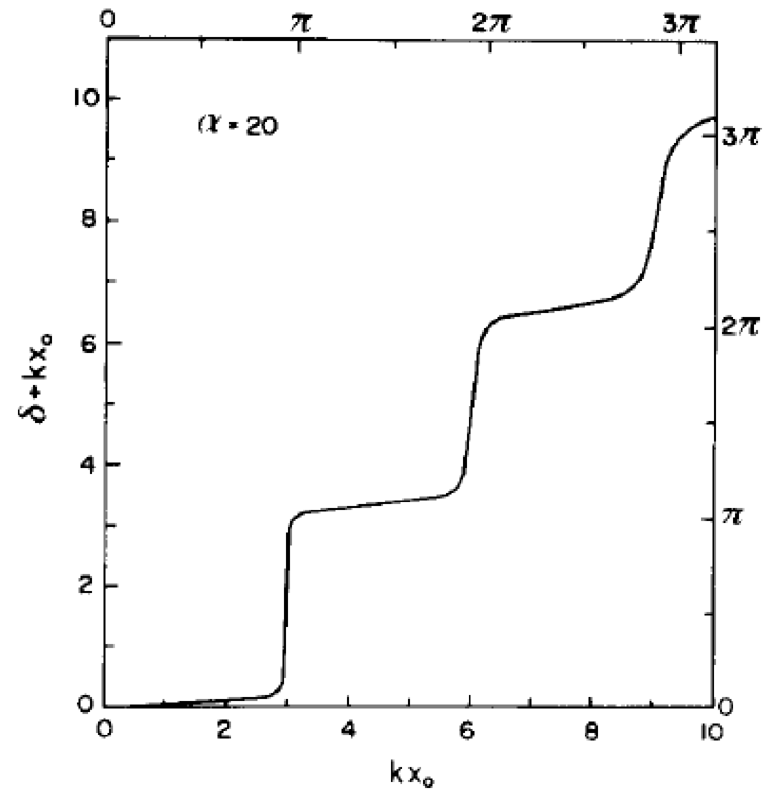
$$\psi_k(x)=\left(\frac{2}{\pi}\right)^{1/2}e^{i\delta_k}\begin{cases}\sin(kx+\delta_k) & x\geqslant x_0\\ A_k\sin(kx) & 0\leqslant x\leqslant x_0\end{cases} \qquad \text{If } W_0\neq 0$$

with

$$\delta_k = -kx_0 + \arctan\left(\frac{\tan(kx_0)}{1 + (\alpha/kx_0)\tan(kx_0)}\right)$$

$$A_k = \{\sin^2(kx_0) + [\cos(kx_0) + (\alpha/kx_0)\sin(kx_0)]^2\}^{-1/2}$$

$$\alpha =: 2\mu x_0 W_0/\hbar^2$$

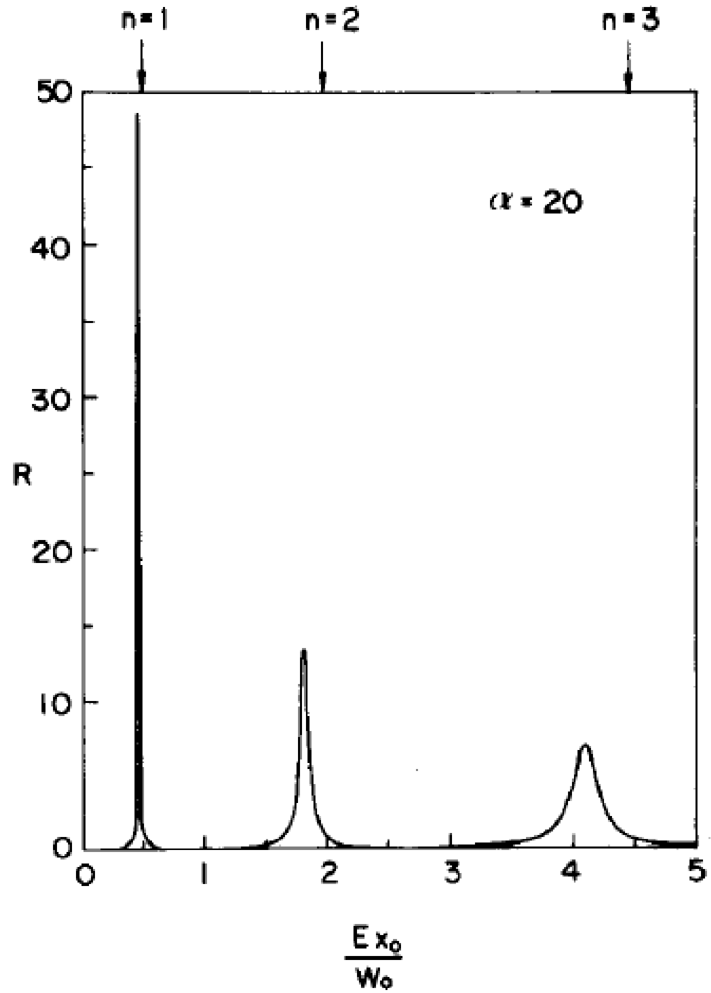


This is the dependence of the phase shift with the wave number; it increases by π giving rise to resonances (kx=nπ)

In this example, the ratio between the probability of finding the particle in the inside region and outside it (in an interval equal to the position of the barrier) is given by

$$R \doteq A_k^2\left(1 - \frac{\sin(2kx_0)}{2kx_0}\right)$$

For a finite value of α



Graph of the ratio R, as a function of the energy

In terms of the S-matrix, the previous results can be written as

$$\psi_k(x) = [i/(2\pi)^{1/2}](e^{-ikx} - e^{2i\delta_k}e^{+ikx})$$

and with the S-matrix defined as

$$S = e^{2i\delta_k}$$

It is found that

$$S = S^{(0)} \cdot e^{2i\arctan(2/\beta)} = S^{(0)}\left(1 - \frac{i}{i/2 - \beta}\right)$$

with

$$S^{(0)} =: e^{-2ikx_0}$$ and

$$\beta =: \frac{1 + (\alpha / kx_0)\tan(kx_0)}{2\tan(kx_0)}$$

Then, if ß->0 we get a resonance, that is the wave number adquires a "resonant value" , such that at the position of the barrier

$$\tan(\bar{k}_n x_0) = -\bar{k}_n x_0 / \alpha$$

For k closer to a resonant value

$$\tan(kx_0) \approx \tan(\bar{k}_n x_0) + x_0(k - \bar{k}_n)$$
$$= -(\bar{k}_n x_0/\alpha) + x_0 \Delta k\,.$$

Then, at lowest order in the difference Δk

$$\beta \approx \frac{\alpha(\alpha+1)\Delta k}{2\bar{k}_n^2 x_0} \approx \frac{\alpha(\alpha+1)\,\mu(E-\bar{E}_n)}{2\hbar^2 \bar{k}_n^3 x_0} = \frac{E-\bar{E}_n}{\Gamma_n}$$

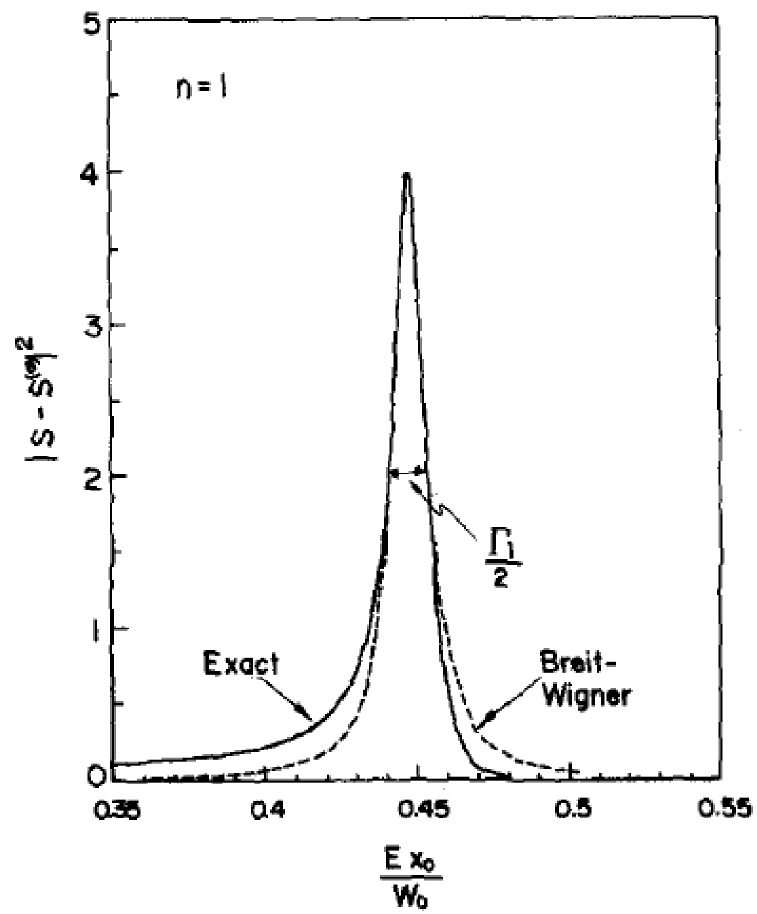
with $\bar{E}_n = \hbar^2 \bar{k}_n^2/(2\mu)$

$$\Gamma_n = \frac{2\hbar^2 \bar{k}_n^3 x_0}{\mu\alpha(\alpha+1)} \underset{\alpha \gg 1}{\approx} \frac{2\hbar^2 \pi^3 n^3}{x_0^2\, \mu\alpha^2}$$

The, for isolated resonances the S-matrix can be written

$$S = S^{(0)}\left(1 - \frac{i\Gamma_n}{(E - \overline{E}_n) + (i/2)\Gamma_n}\right)$$

Thus, we hare obtained again the characteristic Breit-Wigner structure. The agreement between the exact solution and the approximated one, for isolated resonances, is shown in the next figure



. Graph of $|S - S^{(0)}|^2$ as a function of energy for the first resonance.

Another application: Zeno effect

(suggested refs: A. Peres ;Am.J. Phys. 48 (1980) 931; and O.C. and M. Gadella; Phys. Rep. 396(2004)41)

We may say that, from common knowledge, a watched kettle never boils. But if you like to appear as an illustrated girl/man you may say in Quantum Mechanics an unstable system under constant observation will not decay (Zeno's effect)

From the first lecture, we define the non-decay amplitude at the time t

$$A(t) = \langle \psi | \mathrm{e}^{-\mathrm{i}tH} | \psi \rangle = \langle \psi(0) | \psi(t) \rangle$$

and the non-decay probability

$$P(t) = |A(t)|^2 = |\langle \psi | \mathrm{e}^{-\mathrm{i}tH} | \psi \rangle|^2$$

1) for t=0 the derivative of P(t) respecto to t is zero, then for small values of t one has

$$P(t) > \mathrm{e}^{-\gamma t}$$

The proof of this relation, when the eigenvector is outside the domain of the Hamiltonian (like a Gamow vector) is rather lenghty but we can, for the sake of completeness, show a naive version of it, based on the evolution of the system very near a resonance, such that the only significant part of the exponential behavior is given by the imaginary part of the energy. Under this restriction:

$$\left|(\phi, e^{-iHt}\phi)\right|^2 \simeq 1 - (\Delta H)^2 t^2$$

where $\Delta H \approx \Gamma$

hence

$$[1-(\Delta H)^2(t/n)^2]^n > 1-(\Delta H)^2 t^2$$

the left hand side tends to unity as n tends to infinity, it means that if the time interval is small enough (and the measurement is repeated a large number of times during the time interval) the survival amplitude is larger than the exponential and the system does not decay.

2)for intermediate times, starting from a certain time t1, smaller than the half-life, to a certain time t2, larger than the half-life, the non-decay probability is exponential

$$P(t) \approx \mathrm{e}^{-\gamma t}$$

The time t1 needed to begin with the exponential behavior is called the Zeno time.

3) For very large values of t

$$P(t) \approx At^{-n}$$

Alpha decay revisited
(ref: B. Holstein; Am.J.Phys. 64 (1996) 1061)

The alpha-decay rate is expressed

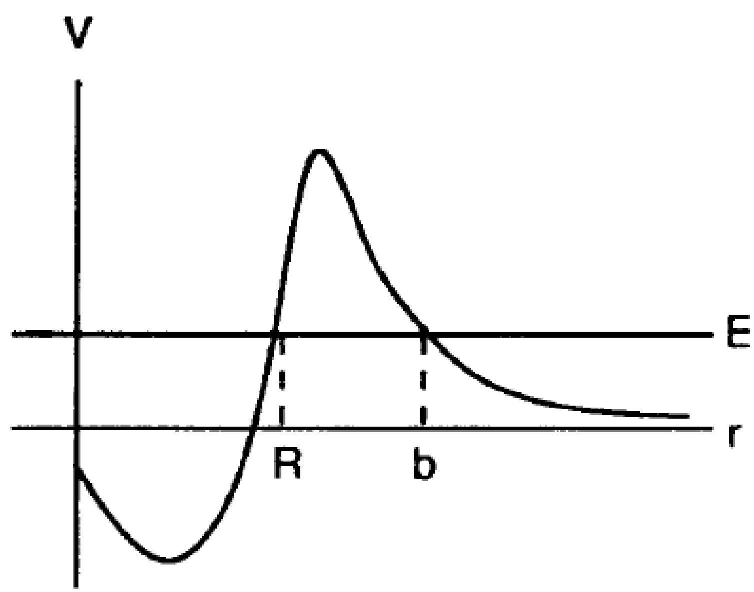
$$\Gamma = \frac{\omega_0}{2\pi} \xi \exp(-2\sigma)$$

where σ is the WKB barrier penetration factor

$$\sigma = \int_R^b dr \sqrt{2M[V(r)-E]}$$

$\omega/2\pi$ is the frequency at which the pre-formed alpha-particle strikes the barrier, and ξ is a factor of the order of unity.

The potential is the sum of the nuclear potential and the Coulomb interaction. In the l=0 wave it looks like



The radial wave functions are of the form

$$\psi(r)=(1/r)u(r)$$

$$u(r)=N\begin{cases}\sin Kr \quad r<R,\\ \dfrac{1}{\sqrt{\kappa(r)}}\left[A\ \exp\left(\int_R^r dr\,\kappa(r')\right)+B\ \exp\left(-\int_R^r dr'\,\kappa(r')\right)\right], \quad R<r<b\\ \dfrac{1}{\sqrt{k(r)}}C\ \exp i\left(\int_b^r dr'\,k(r')-\dfrac{\pi}{4}\right), \quad b<r,\end{cases}$$

With the wave numbers

$$K = \sqrt{2M(E + V_0)}, \quad r < R,$$

$$\kappa(r) = \sqrt{2M[V(r) - E]}, \quad R < r < b$$

$$k(r) = \sqrt{2M[E - V(r)]}, \quad b < r.$$

The relation between the constants, as required by the matching conditios are:

$$C = 2A \exp(\sigma),$$

$$C = iB \exp(-\sigma)$$

Leading to the system

$$\sin KR = \frac{1}{\sqrt{\kappa(R)}}(A+B),$$

$$K \cos KR = \sqrt{\kappa(R)}(A-B)$$

with solutions

$$A = \frac{1}{2}\left(\sqrt{\kappa(R)} \sin KR + \frac{K}{\sqrt{\kappa(R)}} \cos KR \right)$$

$$B = \frac{1}{2}\left(\sqrt{\kappa(R)} \sin KR - \frac{K}{\sqrt{\kappa(R)}} \cos KR \right)$$

but, simultaneously

$$A = \frac{i}{2} B \exp(-2\sigma)$$

Which implies that the energies which may fulfill it must be complex. The associated dispersion relation

$$\tan KR + \frac{K}{\kappa(R)} = \frac{i}{2} \exp(-2\sigma) \left(\tan KR - \frac{K}{\kappa(R)} \right)$$

since the right hand side is suppressed, this equation yields

$$\tan KR \simeq -\frac{K}{\kappa(R)}$$

This is exactly the condition found in the first lecture, in dealing with the definition of Gamow vectors for a single barrier.

Following the conventional WKB treatment (see Holstein) we define the probability flux and the decay rate as

$$\mathbf{S}(r) = \frac{i}{2M}\left(\frac{d\psi^*}{dr}\,\psi - \psi^*\,\frac{d\psi}{dr}\right)\hat{r} = \frac{|C|^2|N|^2}{Mr^2}\,\hat{r}$$

$$\Gamma \approx \frac{4\pi}{M}\,\frac{K^2|N|^2}{\kappa(R)}\exp(-2\sigma)$$

By impossing normalization

$$1 \simeq |N|^2 \int_0^R d^3r \, \frac{1}{r^2} \sin^2 Kr.$$

$$|N|^2 \simeq \frac{1}{2\pi R}$$

The decay rate is re-written

$$\Gamma = \frac{2K^2}{MR\,\kappa(R)} \exp(-2\sigma) = \frac{4K}{\kappa(R)} \frac{\omega_0}{2\pi} \exp(-2\sigma)$$

Naturally, we can understand this result by explicitely writting for the complex eigenvalue

$$E = E_0 - i\Gamma/2$$

in

$$\frac{i}{2}\exp(-2\sigma) = \frac{\dfrac{1}{K}\tan KR + \dfrac{1}{\kappa(R)}}{\dfrac{1}{K}\tan KR - \dfrac{1}{\kappa(R)}}$$

then, we re-obtain, for the wave function in the external region, the result from the previous lectures

$$\psi(r,t)=\frac{u(r)}{r}\,e^{-iEt}$$

$$=\frac{Ce^{-i\pi/4}}{r\sqrt{k(r)}}\,\exp\,i\left(\int_b^r dr' k_0(r')\right)$$

$$\times\exp\left(\frac{\Gamma}{2}\int_b^r dr'\,\frac{M}{k_0(r')}\right)\exp\left(-iE_0t-\frac{\Gamma}{2}\,t\right)$$

The Friedrichs model

The simplest form includes a Hamiltonian with a continuous spectrum (eigenvalues in the real positive axis) plus an eigenvalue embedded in this continuum. The state is represented as

$$\psi = \begin{pmatrix} \alpha \\ \varphi(\omega) \end{pmatrix}$$

α is a complex number and $\varphi(\omega)$ belongs to the space of square integrable functions on the positive real axis

The scalar product is given by

$$\left\langle \begin{pmatrix} \alpha \\ \varphi(\omega) \end{pmatrix} \middle| \begin{pmatrix} \beta \\ \eta(\omega) \end{pmatrix} \right\rangle = \alpha^* \beta + \int_0^\infty \varphi^*(\omega)\eta(\omega)\,\mathrm{d}\omega$$

$$H_0 \psi = \begin{pmatrix} \omega_0 \alpha \\ \omega \varphi(\omega) \end{pmatrix}$$

$$H_0 \begin{pmatrix} 0 \\ \varphi(\omega) \end{pmatrix} = \omega \begin{pmatrix} 0 \\ \varphi(\omega) \end{pmatrix}$$

(restriction of the unperturbed Hamiltonian)

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

State vector for the bound state

$$H = H_0 + \lambda V$$

Total Hamiltonian

$$V\psi = \begin{pmatrix} \int_0^\infty f(\omega)\,\varphi(\omega)\,\mathrm{d}\omega \\ \alpha f^*(\omega) \end{pmatrix}$$

interaction

In order to describe resonances,
we consider the reduced resolvent of H in *the bound state*

$$F_H(z) = \langle 1 | \frac{1}{z-H} | 1 \rangle$$

which is given by

$$\langle 1 | \frac{1}{z-H} | 1 \rangle = \left(-z + \omega_0 + \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z-\omega} \, \mathrm{d}\omega \right)^{-1}$$

$$\eta(z) = z - \omega_0 - \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z - \omega} \, d\omega$$

is a complex analytic function with no singularities on the complex plane other than a branch cut coinciding with the positive semi-axis. It admits analytic continuations. The proof of this result is very technical and we refer the interested reader to the original sources (see Phys. Rep. 396 (2004) 41)

The form of the analytic continuations is

$$\eta_{\pm}(z) = z - \omega_0 - \lambda^2 \int_0^{\infty} \frac{|f(\omega)|^2}{z - \omega \pm \mathrm{i}0} \,\mathrm{d}\omega$$

The poles are determined by the zeros of the equations

$$\eta_{+}(z) = 0 \quad \text{and} \quad \eta_{-}(z) = 0$$

The Friedrichs model in RHS

(formal steps)

a) The spectral decomposition of the unperturbed Hamiltonian

$$H_0 = \omega_0 |1\rangle\langle 1| + \int_0^\infty \omega |\omega\rangle\langle\omega| \, \mathrm{d}\omega$$

b)The decomposition of the interaction

$$V = \int_0^\infty [f^*(\omega)|\omega\rangle\langle 1| + f(\omega)|1\rangle\langle\omega|] \, \mathrm{d}\omega$$

c)The state

$$\psi = \alpha|1\rangle + \int_0^\infty \varphi(\omega)\,|\omega\rangle\,\mathrm{d}\omega$$

d) orthogonality

$$\langle 1|1\rangle = 1 \ ,$$

$$\langle 1|\omega\rangle = \langle \omega|1\rangle = 0 \ ,$$

$$\langle \omega|\omega'\rangle = \langle \omega'|\omega\rangle = \delta(\omega - \omega')$$

e) Scalar product (old notation)

$$\left\langle \begin{pmatrix} 0 \\ \varphi(\omega) \end{pmatrix} \middle| \begin{pmatrix} 0 \\ \eta(\omega) \end{pmatrix} \right\rangle = \int_0^\infty \varphi^*(\omega)\eta(\omega)\,\mathrm{d}\omega$$

Scalar product (new notation)

$$\left\langle \int_0^\infty \varphi^*(\omega)\,\langle\omega|\mathrm{d}\omega \middle| \int_0^\infty \eta(\omega')\,|\omega'\rangle\,\mathrm{d}\omega' \right\rangle = \int_0^\infty \int_0^\infty \varphi^*(\omega)\eta(\omega')\langle\omega|\omega'\rangle\,\mathrm{d}\omega\,\mathrm{d}\omega' \ .$$

f)interaction

$$
\begin{aligned}
V\psi &= \int_0^\infty f^*(\omega)|\omega\rangle\,\mathrm{d}\omega\langle 1|\alpha|1\rangle + \int_0^\infty \int_0^\infty f(\omega)\,\varphi(\omega')|1\rangle\langle\omega|\omega'\rangle\,\mathrm{d}\omega\,\mathrm{d}\omega' \\
&= \alpha \int_0^\infty f^*(\omega)|\omega\rangle\,\mathrm{d}\omega + |1\rangle \int_0^\infty f(\omega)\varphi(\omega)\,\mathrm{d}\omega \ .
\end{aligned}
$$

Here, we intend to get the explicit form of the Gamow vectors for the Friedrichs model.

Let x be an arbitrary positive number (x>0) and write the eigenvalue equation

$$(H - x)\,\Psi(x) = 0$$

with

$$\Psi(x) = \alpha(x)|1\rangle + \int_0^\infty \psi(x,\omega)|\omega\rangle\,\mathrm{d}\omega$$

By applying to it the complete Hamiltonian we get the system of equations

$$(\omega_0 - x)\alpha(\omega) + \lambda \int_0^\infty \psi(x,\omega) f^*(\omega)\,\mathrm{d}\omega = 0$$

$$(\omega - x)\psi(x,\omega) + \lambda f(\omega)\alpha(\omega) = 0\ .$$

It yields an integral equation with one solution of of the form

$$\Psi_+(x) = |x\rangle + \lambda f^*(x)\frac{1}{\eta_+(x)}\left\{|1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{x-\omega+\mathrm{i}0}|\omega\rangle\,\mathrm{d}\omega\right\}$$

This is a functional in the right-sail space of the RHS. It admits a continuation to the lower half plane, with a single pole:

$$\Psi_+(z) = \frac{C}{z - z_0} + o(z)$$

then

$$0 = (H - z)\Psi_+(z) = \frac{1}{z - z_0}(H - z)C + (H - z)o(z)$$

which gives

$$(H - z_0)C = 0 \Rightarrow HC = z_0 C$$

then

$$\Psi_+(z) \approx \frac{\text{constant}}{(z - z_0)} \left\{ |1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{z - \omega + \mathrm{i}0} |\omega\rangle \, \mathrm{d}\omega \right\}$$

with the help of Taylor's theorem

$$\frac{1}{z-\omega+\mathrm{i}0}=\frac{1}{z_0-\omega+\mathrm{i}0}-\frac{z-z_0}{(z_0-\omega+\mathrm{i}0)^2}+o(z)$$

we get

$$\Psi_+(z)\approx\frac{\text{constant}}{(z-z_0)}\left\{|1\rangle+\lambda\int_0^\infty\frac{f(\omega)}{z_0-\omega+\mathrm{i}0}\,|\omega\rangle\,\mathrm{d}\omega\right\}$$

thus

$$C = |f_0\rangle = |1\rangle + \int_0^\infty \frac{\lambda f(\omega)}{z_0 - \omega + \mathrm{i}0} \, |\omega\rangle \, \mathrm{d}\omega$$

This is the formal solution of the decaying Gamow vector for the Friedrichs model . In the same way, we have for the growing Gamow vector the expression

$$|\tilde{f}_0\rangle = |1\rangle + \int_0^\infty \frac{\lambda f^*(\omega)}{z_0^* - \omega - \mathrm{i}0} \, |\omega\rangle \, \mathrm{d}\omega$$

Some applications to nuclear structure calculations

-Eigenvalue problem for Gamow vectors in N-N interactions
Refs: M. Baldo, L. S. Ferreira and L. Streit (Phys. Rec. C. 36 (1987) 1743

- Two particle resonant states in a many body mean field

Refs: R. J. Liotta et al; Phys. Rev. Lett 89 (2002) 042501

- The Friedrichs model with fermion boson couplings

Refs: O.C, M. Gadella and G. Pronko
(J.Mod.Phys.E 15 (2006) 1273; ibid. 16 (2007)1.

The extended Friedrichs model

(a fermion interacting with a boson field which has a discrete and a resonant state

$$H_I = \omega_0 |1\rangle\langle 1| + \int_0^\infty d\omega \omega |\omega\rangle\langle\omega| + \sum_k c_k |k\rangle\langle k|$$

$$H_{II} = \sum_{k,l} \left[h_{k,l} |k,1\rangle\langle l| + h^*_{k,l} |l\rangle\langle k,1| \right]$$

$$H_{III} = \sum_{k,l} \int_0^\infty d\omega \left[f_{k,l}(\omega) |k,\omega\rangle\langle l| + f^*_{k,l}(\omega) |l\rangle\langle k,\omega| \right]$$

$$H_{IV} = \sum_{k,k'} \int_0^\infty d\omega \left[g_{kk'}(\omega) |k,1\rangle\langle k',\omega| + g^*_{kk'}(\omega) |k',\omega\rangle\langle k,1| \right]$$

$$\sum_l f^*_{kl}(\omega) h_{lk'} = g_{kk'}(\omega)\, \delta_{kk'} = g_k(\omega)\, \delta_{kk'}$$

$$(H - E)\Psi(E) = 0$$

$$\Psi(E) = \sum_k \varphi_k(E)|k\rangle + \sum_k \phi_{k,1}(E)|k,1\rangle + \sum_k \int_0^\infty d\omega\, \psi_k(E,\omega)|k,\omega\rangle$$

$$
\begin{aligned}
(H-E)\Psi(E) \equiv & \sum_k \varphi_k(E)(c_k - E)|l\rangle + \sum_{k,l} h^*_{kl}\phi_{k,1}(E)|k\rangle \\
+ & \sum_k (c_k + \omega_0 - E)\phi_{k,1}(E)|k,1\rangle + \sum_k \int_0^\infty d\omega \psi_k(E,\omega) g_k(\omega)|k,1\rangle \\
+ & \sum_k \phi_{k,1}(E)\int_0^\infty d\omega g^*_k(\omega)|k,\omega\rangle + \sum_k \int_0^\infty d\omega \psi_k(E,\omega)(c_k+\omega-E)|k,\omega\rangle \\
+ & \sum_{k,l}\int_0^\infty d\omega \psi_k(E,\omega) f^*_{kl}(\omega)|l\rangle + \sum_{i,k}\varphi_k(E)h_{ik}|i,1\rangle
\end{aligned}
$$

$$
+ \sum_{i,k}\int_0^\infty d\omega \varphi_k(E) f_{ik}(\omega)|i,\omega\rangle = 0
$$

which leads to the coupled system

$$
\psi_k(E,\omega) = c\delta(c_k - \omega - E) - \sum_l \frac{\varphi_l(E) f_{kl}(\omega)}{c_k - \omega - E} - \frac{\phi_{k1}(E) g^*_k(\omega)}{c_k - \omega - E}
$$

One can introduce each of these fields in the original equations, to re-write the coupled system

$$\left[(c_k - E)\delta_{km} \quad - \quad \sum_m A_{km}(E)\right] \varphi_m(E)$$

$$+\sum_m (h^*_{mk} \quad - \quad B_{km}(E))\,\phi_{m1}(E) = -c\sum_m f^*_{mk}(E - c_m)$$

$$\sum_l [h_{kl} - \widetilde{B}_{kl}(E)]\varphi_l(E)$$

$$+(c_k + \omega_0 - E - C_k(E))\phi_{k1}(E) = -cg_k(E - c_k)$$

leading to the following system of equations

$$\varphi_k(E)(c_k - E) + \sum_l h^*_{lk}\phi_{l1}(E) + \sum_l \int_0^\infty d\omega \psi_l(E,\omega) f^*_{lk}(\omega) = 0$$

$$\sum_l \varphi_l(E) h_{kl} + (c_k + \omega_0 - E)\phi_{k1}(E) + \int_0^\infty d\omega \psi_k(E,\omega) g_k(\omega) = 0$$

$$\psi_k(E,\omega)(c_k + \omega - E) + \sum_l \varphi_l(E) f_{kl}(\omega) + \phi_{k1}(E) g^*_k(\omega) = 0$$

With the quantities A, B, etc, given by

$$A_{km}(E) = \int_0^\infty d\omega \, \sum_l \frac{f^*_{lk}(\omega)\, f_{lm}(\omega)}{c_l + \omega - E}$$

$$B_{km}(E) = \int_0^\infty d\omega \, \frac{f^*_{mk}(\omega)\, g^*_m(\omega)}{c_m + \omega - E}$$

$$\widetilde{B}_{km}(E) = \int_0^\infty d\omega \, \frac{g_k(\omega)\, f_{km}(\omega)}{c_k + \omega - E}$$

$$C_k(E) = \int_0^\infty d\omega \, \frac{|g_k(\omega)|^2}{c_k + \omega - E}$$

Then, on formal grounds, we have shown that the coupling of a bound fermion state with a boson field with has a resonant state yields a resonant behavior for the fermion.

This finding has some significance for nuclear structure calculations, where, like in Berggreen's basis, the fermions include Gamow resonances and they are used to construct boson fields (like two-particle or particle-hole excitations) some applications of the Friedrisch model to physical systems (Onley and Kumar; Am. J. Phys. 60 (19929 431:

We shall consider a system which consists of a particle wich can exists in a discrete state (particle-hole state) or in a continuum as a free massive boson, then applying our previous concepts we write the wave function of the free particle as

$$\Psi = a(t)\psi_1\bar{\psi}_2 + \int d^3k\, b(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{r}}$$

The interaction between both sectors of the wave function can be written

$$(H_{\text{int}})_{0k} = \int d^3r\, \psi_1 \bar{\psi}_2^* g e^{i\mathbf{k}\cdot\mathbf{r}} e^{i(\omega_0 - \omega_k)t}$$

$$= V(\mathbf{k}) e^{i(\omega_0 - \omega_k)t},$$

The system is prepared at t=0 in its discrete state, in the particle-hole state, with the initial conditions

$$a(0) = 1, \quad b(\mathbf{k},0) = 0.$$

The equation of motion (in the interaction picture)

$$i\frac{\partial\Psi}{\partial t} = H_{\text{int}}\Psi,$$

Yields the equations for the amplitudes

$$i\frac{da}{dt} = \int d^3k\, V(\mathbf{k}) b(\mathbf{k},t) e^{i(\omega_0 - \omega_k)t},$$

$$i\frac{\partial b}{\partial t} = V^*(\mathbf{k}) a(t) e^{i(\omega_k - \omega_0)t}.$$

The, from both equations one gets

$$\frac{da}{dt} = -\int d^3k \, |V(\mathbf{k})|^2 \int_0^t dt' \, a(t') e^{i(\omega_0 - \omega_k)(t - t')}.$$

After a long time (t->∞) b(k,∞) is the probability amplitude to find the boson state and a(∞) is the survival amplitude in the p-h state. If a(t<0)=0, its Fourier transform can be written

$$f(\omega) = \frac{1}{2\pi} \int_0^\infty dt \, a(t) e^{i\omega t}$$

and

$$a(t) = \int_{-\infty}^{\infty} d\omega\, f(\omega) e^{-i\omega t},$$

$$b(\mathbf{k}, \infty) = -2\pi i V^*(\mathbf{k}) f(\omega_k - \omega_0).$$

To get the expression for f(ω) we integrate

$$\int_0^{\infty} dt\, e^{(i\omega - \epsilon)t} \frac{da}{dt}$$

$$= -\int d^3k\, |V(\mathbf{k})|^2 \int_0^{\infty} dt\, e^{[(i\omega - \epsilon) + i(\omega_0 - \omega_k)]t}$$

$$\times \int_0^{t} dt'\, a(t') e^{-i(\omega_0 - \omega_k)t'},$$

and take $\epsilon \to 0$

therefore

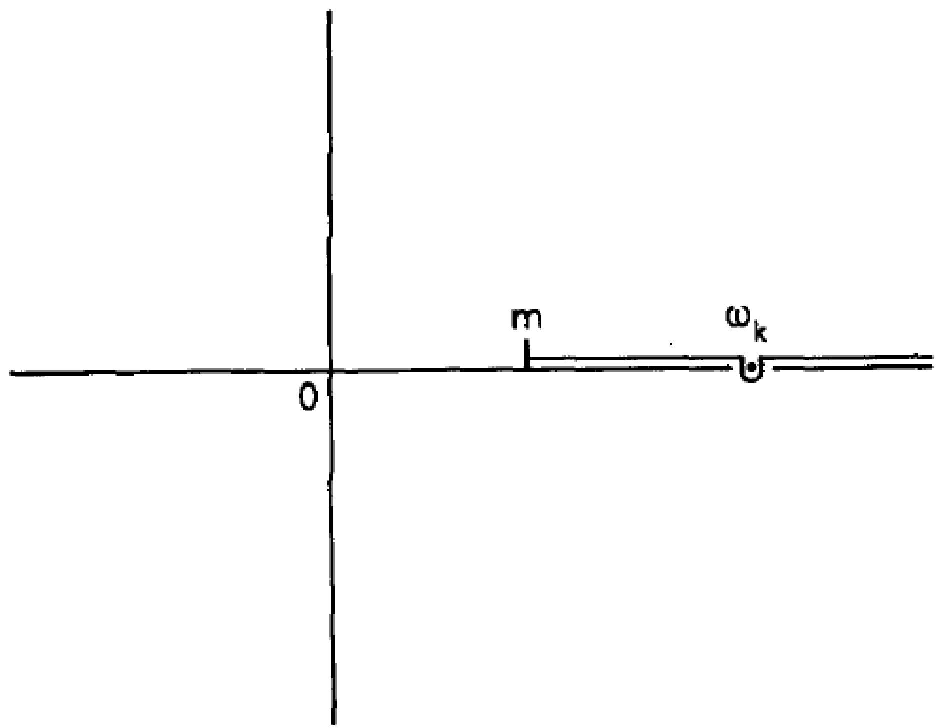
$$f(\omega) = \left[2\pi i\left(-\omega + \int d^3k' \frac{|V(\mathbf{k}')|^2}{\omega + \omega_0 - \omega_{k'} + i\epsilon}\right)\right]^{-1}.$$

This integral is transformed as an integral on complex energies

$$\mathbf{Z}(\omega_k) = \int d^3k' \frac{|V(\mathbf{k}')|^2}{\omega_k - \omega_{k'} + i\epsilon}$$

$$= \int_m^\infty d\omega_{k'} \frac{\rho(\omega_{k'})|V(\mathbf{k}')|^2}{(\omega_k + i\epsilon) - \omega_{k'}}$$

In the complex energy plane the integration path is of the form



since

$$\int_{\text{path 1}} d\omega_{k'} \frac{Q(\omega_{k'})}{\omega_k - \omega_{k'}} = P\int d\omega_{k'} \frac{Q(\omega_{k'})}{\omega_k - \omega_{k'}} - \pi i Q(\omega_k).$$

$$\Omega(\omega_k) = P\int_m^\infty d\omega_{k'} \frac{|V(\mathbf{k}')|^2 \rho(\omega_{k'})}{\omega_k - \omega_{k'}},$$

$$Q(\omega_k) = \rho(\omega_k)|V(\mathbf{k})|^2.$$

Since now we have an explicit expression for f(ω) we can write, for the survival amplitude

$$\text{for } t > 0, \quad a(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{Z(\omega) - \omega + \omega_0}.$$

Which may be solved, by integration, for an explicit structure of Z(ω)

The proposed dependence of the interaction and of the density of states is

$$|V(k)|^2\rho(\omega_k) = g^2[(\omega_k^2 - m^2)/(\omega_k^2 - b^2)^2]$$

Where g is a coupling strength factor, m is the mass of the boson and b is a real value whose absolute value is smaller than the mass m

Under this assumption, the integral of Z yields

$$\mathbf{Z}(\omega) = g^2\left(\frac{b^2 + \omega m}{2b^2(b^2 - \omega^2)} + \frac{(2b - \omega)m^2 - b^2\omega}{4b^3(b - \omega)^2}\right.$$
$$\times \ln(m - b) + \frac{(2b + \omega)m^2 + b^2\omega}{4b^3(b + \omega)^2}\ln(m + b)$$
$$\left. + \frac{\omega^2 - m^2}{(b^2 - \omega^2)^2}\ln(m - \omega)\right).$$

$$Z(\omega) = \Omega(\omega) + iQ(\omega)\arg(m-\omega),$$

$$-\frac{3\pi}{2} < \arg(m-\omega) < \frac{\pi}{2},$$

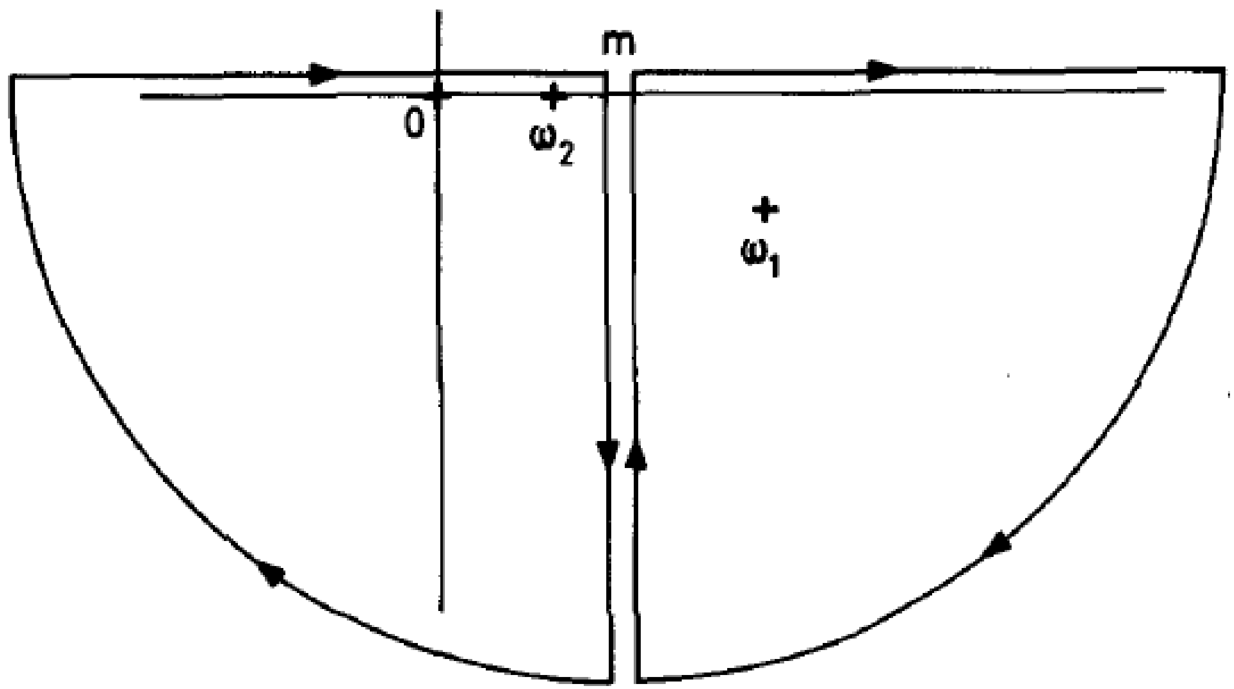
$$\Omega(\omega) = g^2\left(\frac{b^2+\omega m}{2b^2(b^2-\omega^2)} + \frac{(2b-\omega)m^2 - b^2\omega}{4b^3(b-\omega)^2}\right.$$

$$\times \ln(m-b) + \frac{(2b+\omega)m^2+b^2\omega}{4b^3(b+\omega)^2}\ln(m+b)$$

$$\left. + \frac{\omega^2-m^2}{(b^2-\omega^2)^2}\ln|m-\omega|\right),$$

$$Q(\omega) = g^2[(\omega^2-m^2)/(\omega^2-b^2)^2].$$

And with it, we integrate the expression of the survival amplitude in the regions limited by the contour



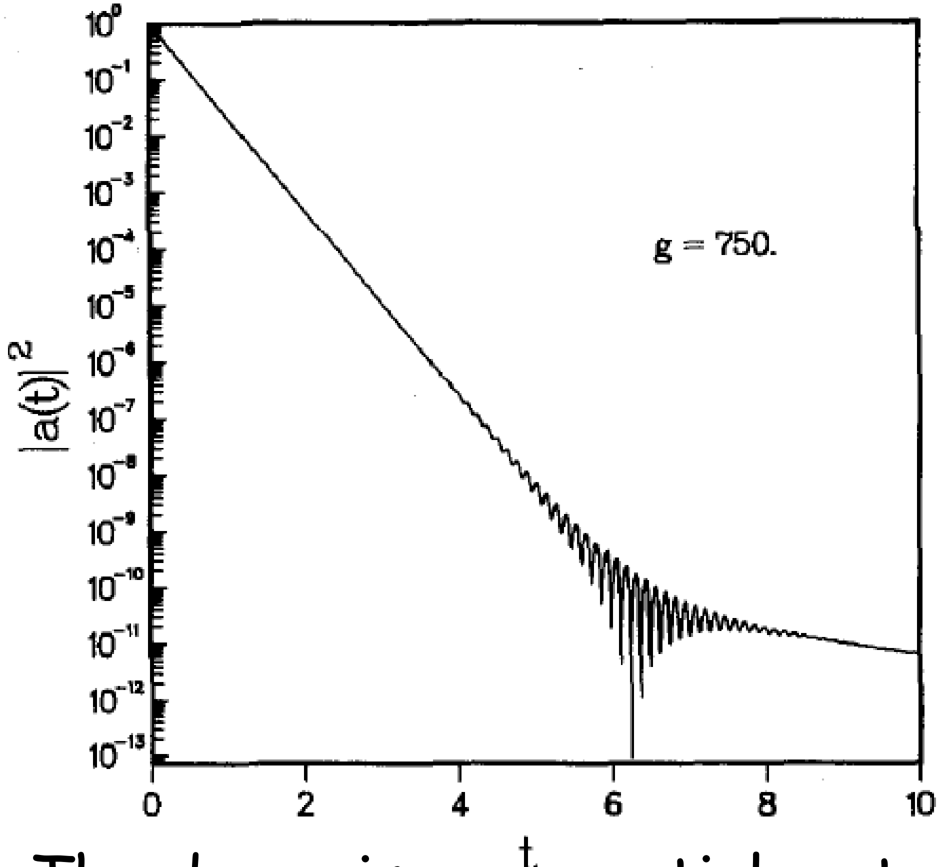
$$a(t) = -\sum_{\text{poles},j} R_j e^{-i\omega_j t} + \frac{1}{2\pi i}\int_m^{m-i\infty} d\omega \left(\frac{1}{Z(\omega) - \omega + \omega_0}\right)_{\omega-\epsilon}^{\omega+\epsilon} e^{-i\omega t},$$

The contribution from the cut is a slowly varying function of the form

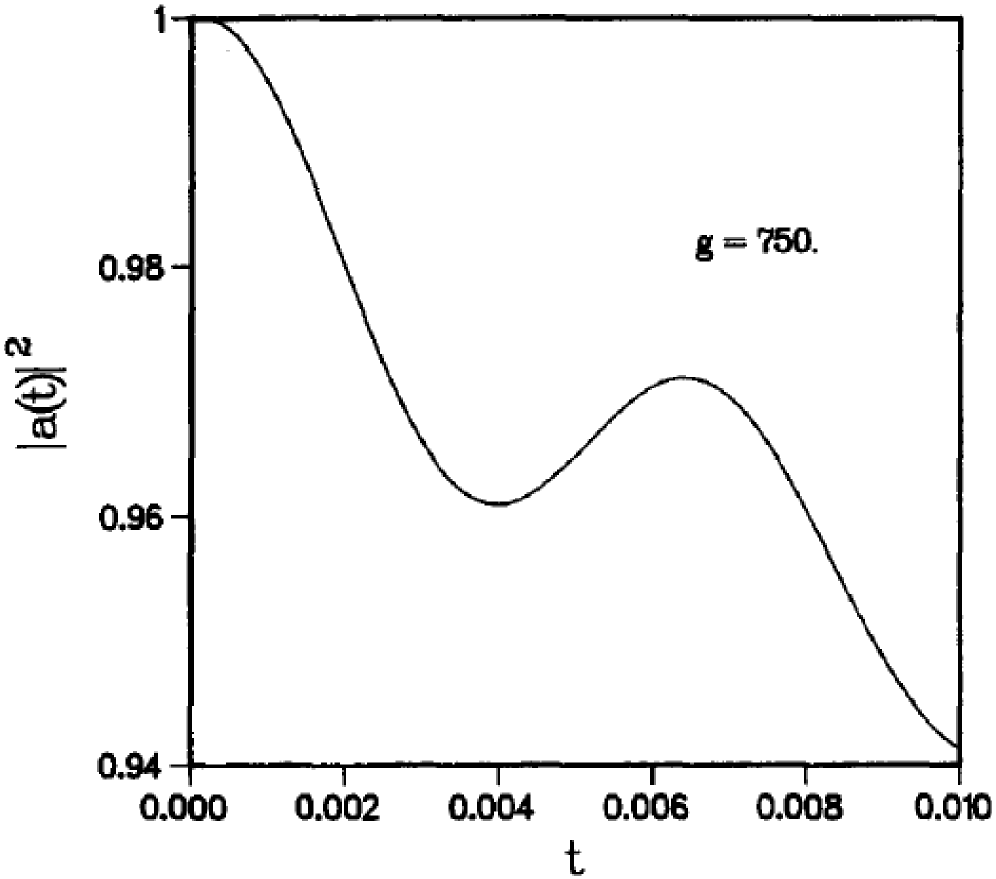
$$-ie^{-imt}\int_0^\infty dy \frac{Q(m-iy)e^{-yt}}{[\Omega(m-iy) - \frac{1}{2}i\pi Q(m-iy) - m + iy + \omega_0]^2 + [\pi Q(m-iy)]^2}$$

The following are results given by Onley and Kumar, and they illustrated rather nicely the transition from the linear to the exponential decay

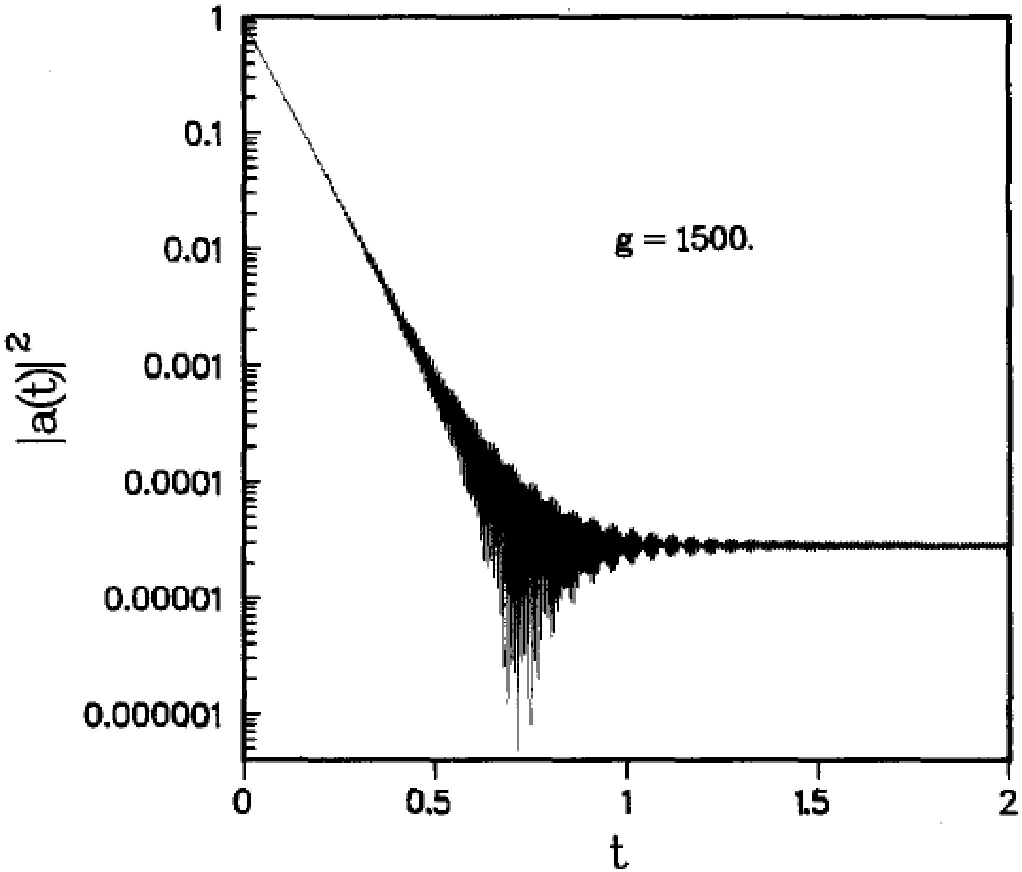
$$\omega_0 = 1000\ \text{MeV}, \quad m = 300\ \text{MeV},$$
$$a(=\sqrt{m^2 - b^2}) = 50\ \text{MeV},$$
$$g = 750\text{–}4000\ \text{MeV}^{3/2},$$



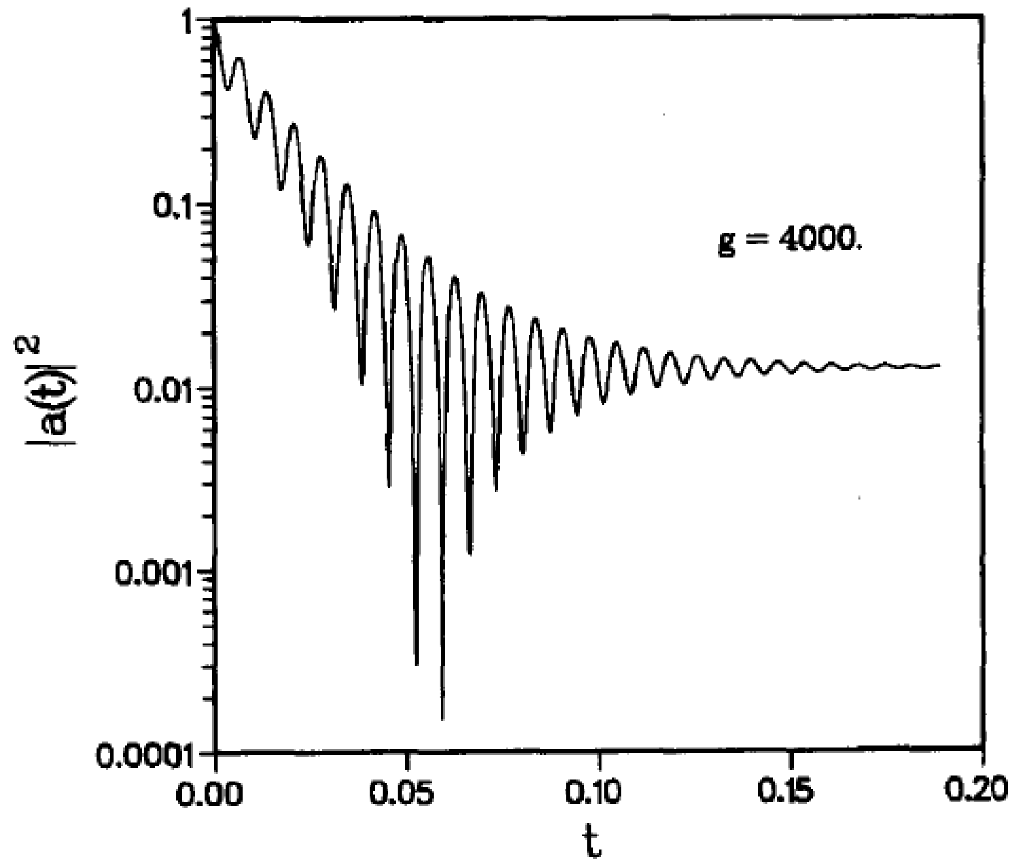
The decay is exponential up to 5 lifetimes (the time is given in units of the lifetime), but the regime changes at about 6 lifetimes due to interferences between the pole and the cut



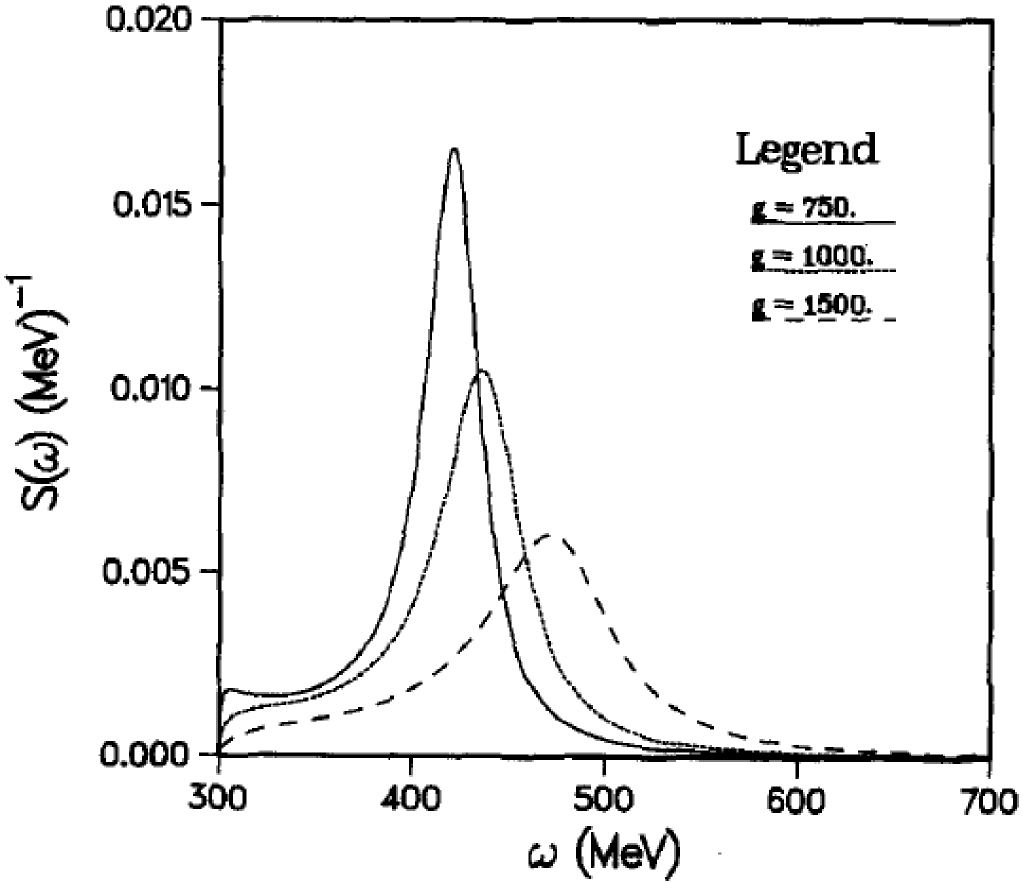
Survival for small fractions of the lifetime



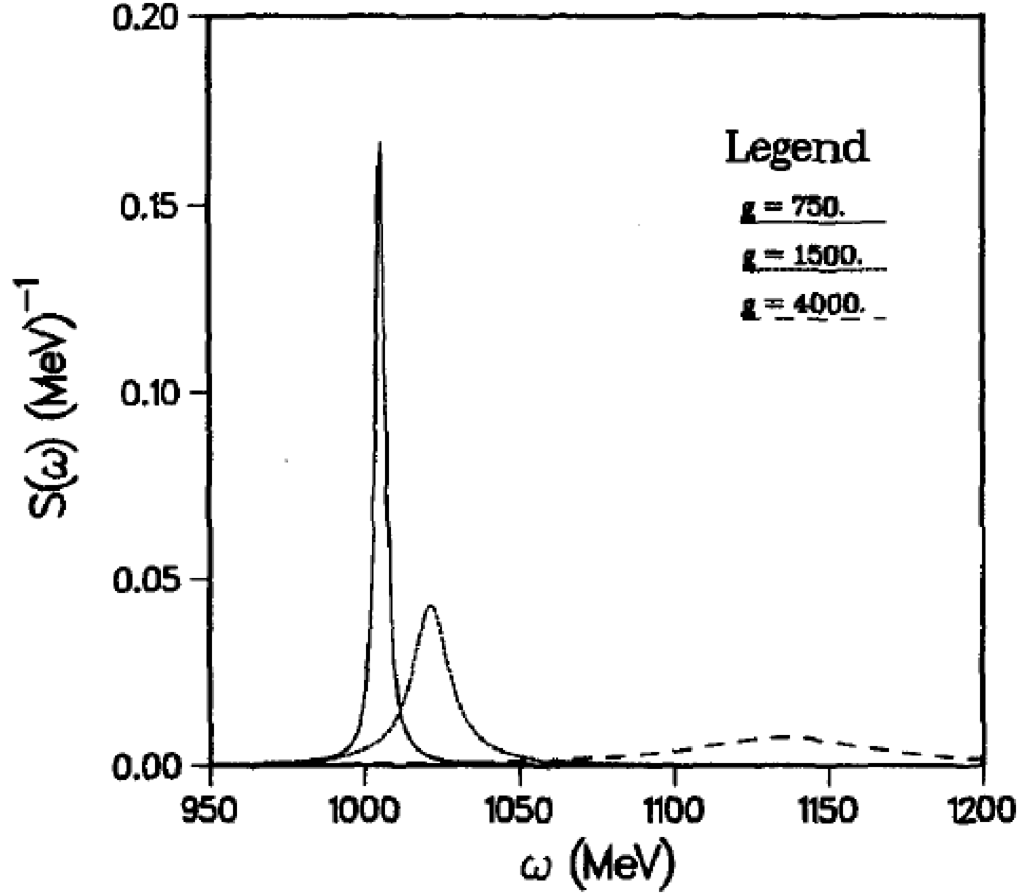
Dependence of the survival upon the coupling constant



Onset of non-exponential decay, for the strong coupling case



Line shapes (for the continuous part of the spectrum, for w_0 close to m



Line shapes (for the continuous part of the spectrum, for w_0 1000 MeV

Assisted tunneling of a metastable state between barriers (G. Kalbermann; Phys. Rev. C. 77 (2008) 041601 (R)

Problem: exposing a system to external excitations can enhance its tunneling rate.

Model:

$$i\,\frac{\partial \Psi}{\partial t} = \frac{-1}{2\,m}\frac{\partial^2 \Psi}{\partial x^2} + \lambda\,(\delta(x+x_0) + \delta(x-x_0))\Psi$$

Initial non-stationary state

$$\Psi(x, t=0) = N e^{-\frac{x^2}{\Delta^2}}$$

Stationary states (even and odd solutions)

$$n_e(k)\chi_e(x) = \begin{cases} \cos(kx) & \text{if} \quad |x| < x_0 \\ A\cos(kx) \pm B\sin(kx) & \text{if} \quad x > x_0 \text{ or } x < -x_0, \end{cases}$$

$$n_o(k)\chi_o(x) = \begin{cases} \sin(kx) & \text{if} \quad |x| < x_0 \\ \pm C\cos(kx) + D\sin(kx) & \text{if} \quad x > x_0 \text{ or } x < -x_0. \end{cases}$$

Normalization factors (even solutions)

$$(n_e(k))^2 = \pi(A(k)^2 + B(k)^2),$$

$$A(k) = 1 - \sin(2\,k\,x_0)\frac{m\lambda}{2k},$$

$$B(k) = \frac{m\lambda}{k}(\cos(kx_0))^2,$$

$$k^2(n_e(k))^2 = \pi\left(\left(k - \sin(2\,k\,x_0)\frac{m\lambda}{2}\right)^2 + (m\lambda(\cos(kx_0))^2)^2\right)$$

Normalization factors (odd solutions)

$$(n_o(k))^2 = \pi(C(k)^2 + D(k)^2),$$

$$C(k) = -\frac{m\lambda}{k}(\sin(kx_0))^2,$$

$$D(k) = 1 + \sin(2\,k\,x_0)\frac{m\lambda}{2k},$$

$$k^2(n_o(k))^2 = \pi\left(\left(k + \sin(2\,k\,x_0)\frac{m\lambda}{2}\right)^2 + (m\lambda(\sin(kx_0))^2)^2\right)$$

External perturbation

$$V(x,t) = \mu x \sin(\omega t)$$

Wave function

$$\Psi(x,t) = \sum_{i=e,o} \int_0^\infty \chi_i(k,x) a_i(k,t) e^{\frac{-ik^2 t}{2m}} dk$$

Amplitudes:

$$i\dot{a}_e(k) = \int e^{-\frac{(k'^2-k^2)t}{2m}} \times \langle \chi_e(k,x)|V(x,t)|\chi_o(k',x)\rangle dk' a_o(k',t),$$

$$i\dot{a}_o(k) = \int e^{-\frac{(k'^2-k^2)t}{2m}} \times \langle \chi_o(k,x)|V(x,t)|\chi_e(k',x)\rangle dk' a_e(k',t).$$

Matrix element of the external potential

$$
\begin{aligned}
&\langle \chi_e(k,x)|V(x,t)|\chi_o(k',x)\rangle \\
&\quad = \int_{-\infty}^{\infty} \chi_e(k,x)V(x,t)\chi_o(k',x)dx \\
&\quad \approx \mu\pi \frac{1}{n_e(k)n_o(k')} \frac{\partial\delta(k-k')}{\partial k'} \sin(\omega t)
\end{aligned}
$$

Then, the eqs. for the amplitudes read

$$i\dot{a}_e(k,t) = -\frac{\pi}{n_o(k)n_e(k)}\Big(a_o'(k,t) - n_o'(k)/n_o(k)a_o(k,t) - \frac{ikt}{m}a_o(k,t)\Big)\mu\sin(\omega t),$$

(for even solutions)

$$i\dot{a}_o(k,t) = -\frac{\pi}{n_o(k)n_e(k)}\Big(a_e'(k,t) - n_e'(k)/n_e(k)a_e(k,t) - \frac{ikt}{m}a_e(k,t)\Big)\mu\sin(\omega t)$$

(for odd solutions)

Rescaling the variables:

$$k \to \tilde{k} = \frac{k}{\sqrt{m\omega}},$$
$$t \to \tilde{t} = \omega t,$$
$$\mu \to \tilde{\mu} = \frac{\mu}{\sqrt{\omega^3 m}},$$

The amplitudes are given by

$$\begin{aligned} i\dot{a}_e(k,\tilde{t}\,) = -\tilde{\mu}\frac{\pi}{n_o(k)n_e(k)}&(a_o'(k,\tilde{t}\,) - n_o'(k)/n_o(k)a_o(k,\tilde{t}\,) \\ &- ik\tilde{t}a_o(k,\tilde{t}\,))\sin(\tilde{t}\,), \end{aligned}$$

$$\begin{aligned} i\dot{a}_o(k,\tilde{t}\,) = -\tilde{\mu}\frac{\pi}{n_o(k)n_e(k)}&(a_e'(k,\tilde{t}\,) \\ &- n_e'(k)/n_e(k)a_e(k,\tilde{t}\,) - ik\tilde{t}a_e(k,\tilde{t}\,))\sin(\tilde{t}\,). \end{aligned}$$

The normalization factors have zeros at the values

$$k^2 n_{e,o}^2 \approx \gamma_{e,o}^{(j)} \left(k^2 - k_{e,o}^{(j)2}\right)^2 + \beta_{e,o}^{(j)},$$

where j is the pole index, with

$$k_e = \frac{(2n+1)\pi m\lambda}{2(1+m\lambda x_0)}, \quad k_o = \frac{n\pi m\lambda}{(1+m\lambda x_0)},$$

$$\gamma_e = \pi \frac{2m\lambda x_0^3(m\lambda x_0 + 4)}{(2n+1)^2\pi^2}, \quad \gamma_o = \pi \frac{m\lambda x_0^3(m\lambda x_0 + 4)}{4n^2\pi^2}$$

$$\beta_e = \frac{(2n+1)^4\pi^4}{16m^2\lambda^2 x_0^4}, \quad \beta_o = \frac{n^4\pi^4}{m^2\lambda^2 x_0^4}.$$

with these expressions the poles are located symmetrically above and below the real momentum axis, with values given by

$$q_n^e = \left(\frac{(2n+1)\pi m\lambda}{2(1+m\lambda x_0)}\right)^2 \pm i\sqrt{\frac{\beta_e}{\gamma_e}},$$

$$q_o^2 = \left(\frac{n\pi m\lambda}{(1+m\lambda x_0)}\right)^2 \pm i\sqrt{\frac{\beta_o}{\gamma_o}}.$$

In Kalbermann's calculations, the pole structure is of the form:

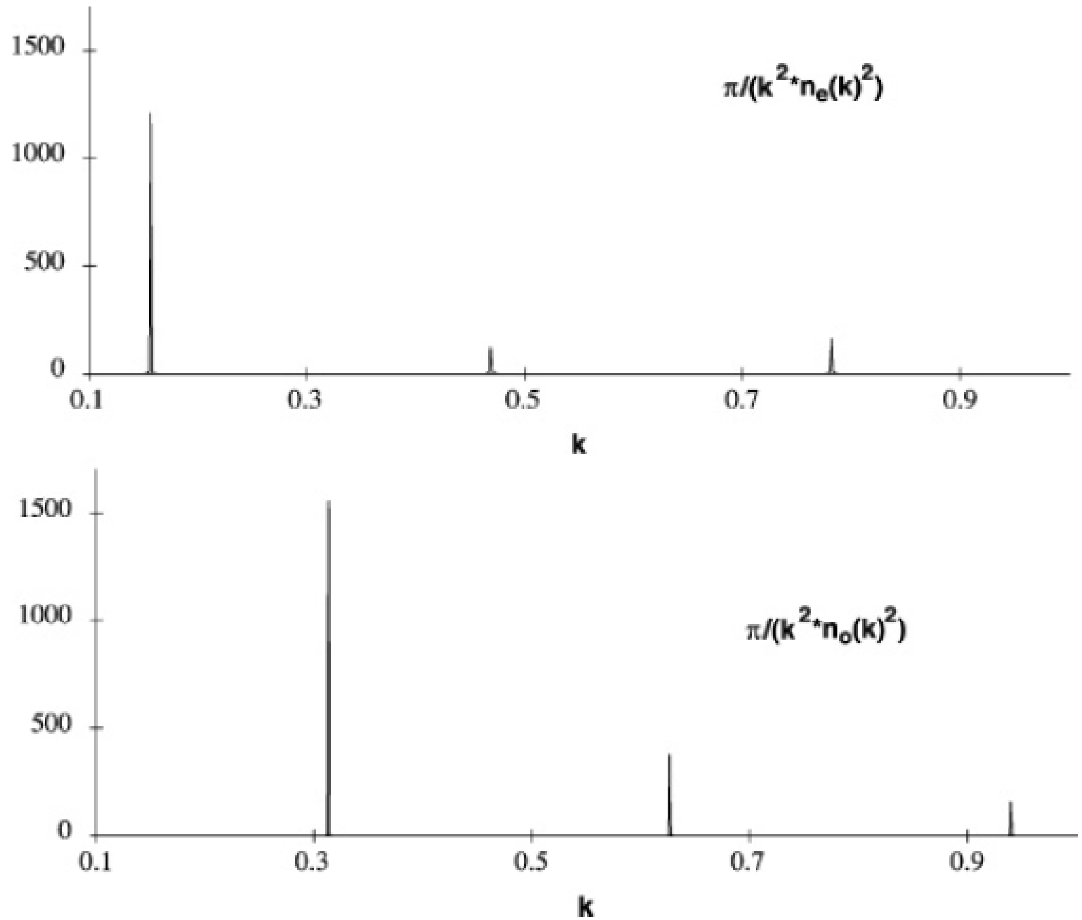


FIG. 1. $\frac{\pi}{k^2 n_e(k)^2}$ and $\frac{\pi}{k^2 n_p(k)^2}$ as a function of k in units of fm^{-1} for the parameters $m\lambda x_0 = 400$, $x_0 = 10$ fm.

The first pole dominates the structure of the wave function, as of the decay constant, which is then given by (even states)

$$|e^{-i\frac{k^2_{e,n=1}t}{2m}}| \to e^{-\Lambda_e t}$$

$$\Lambda_e = \frac{1}{2m}\sqrt{\frac{\beta_1}{\gamma_1}} = \frac{\pi^3}{8mx_0^4m^2\lambda^2},$$

And (odd states) $\Lambda_o = 8\Lambda_e.$

Then, from Kalbermann's results it is seen that two main effects are due to the action of the external potential upon the decaying states, namely:

a) The decay constant is enhanced
b) The dominance of the first pole produces an extra damping

That is: the external field causes the acceleration of the tunneling (the wave function will tunnel faster)

Two-Particle Resonant States in a Many-Body Mean Field (from PRL 89 (2002) 042501

"…The role played by single-particle resonances and of the continuum itself upon particles moving in the continuum of a heavy nucleus is not fully understood.
One may approach this problem by using, as the single-particle representation, the Berggren representation.
One chooses the proper continuum as a *given* contour in the complex energy plane and forms the basis set of states as the bound states plus the Gamow resonances included in that contour plus the scattering states on the contour…" (quoted from the paper)

One may
thus think that the Berggren representation can also be used straightaway to evaluate many-particle quantities, as one does with the shell model using bound representations. Unfortunately this is not the case. The root of the problem is that the set of energies of the two-particle basis states may cover the whole complex energy plane of interest…."

TABLE I. Energies (in MeV) of the $\lambda = 0$ first excited bound state and of the lowest two-particle resonances in ^{80}Ni. The energies are given as a function of the number of scattering states included in the single-particle representation, i.e., the number N_g of Gaussian points. For $N_g = 0$ the representation consists of bound states and Gamow resonances only. The calculation of these states was performed by using the high precision piecewise perturbation method [6].

N_g	E_1	E_2	E_3	E_4
0	(−0.642, 0.012)	(2.158, 0.719)	(3.268, −0.883)	(7.931, −0.198)
35	(−0.654 17, 0)	(1.968 74, −0.392 35)	(3.924 20, −1.052 08)	(7.956 93, −0.252 36)
70	(−0.652 74, 0)	(1.969 88, −0.393 21)	(3.924 29, −1.051 59)	(7.956 91, −0.252 51)
110	(−0.652 74, 0)	(1.972 61, −0.398 38)	(3.924 16, −1.051 68)	(7.956 87, −0.252 50)
225	(−0.653 08, 0)	(1.972 41, −0.399 35)	(3.923 90, −1.051 89)	(7.956 85, −0.252 49)
550	(−0.653 08, 0)	(1.972 41, −0.399 35)	(3.923 90, −1.051 89)	(7.956 85, −0.252 49)

Then, the main contribution to the imaginary part of the energy of the many-body states is given by Gamow vectors in the single-particle basis, as expected, with no effects coming from scattering states. To illustrate this point we go to the nex example (dispersion of neutrinos by a nucleus)

Some applications to nuclear structure and nuclear reactions

refs: O.C, R. J. Liotta and M. Mosquera
(Phys. Rev. C. 78 (2008) 064308)

We study the process (charge current neutrino-nucleos scattering)

$$\nu + ^{208}\mathrm{Pb} \rightarrow \mathrm{e}^- + ^{208}\mathrm{Bi}^*$$

By assuming that the single particle states to be used in the calculations belong to a basis with bound, quasi-bound, resonant and scattering states.

The proton-particle states are (al large distances they behave as exp{ikr}):

a) bound states, for which Re(k)=0, Im(k)>0,

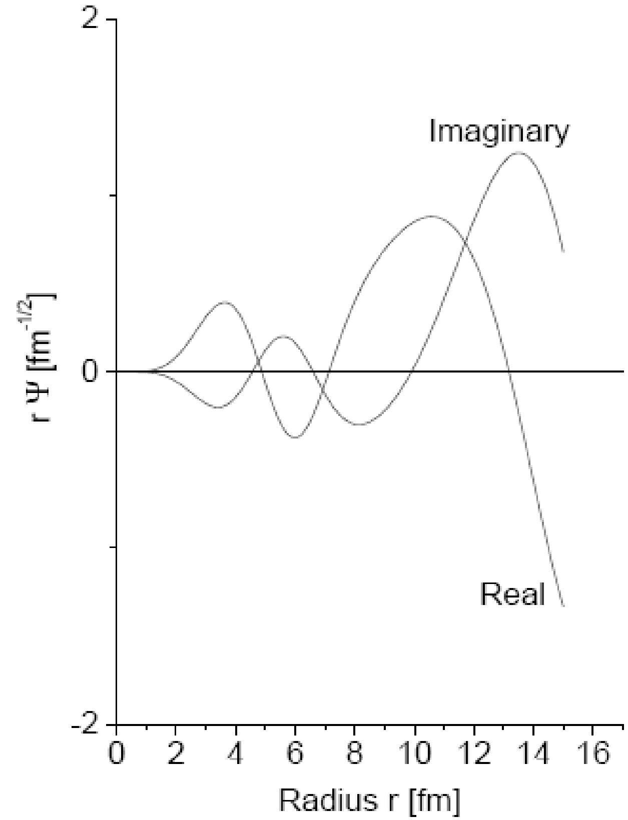
b)anti-bound states, for which Re(k)=0, Im(k)<0.

c) outgoing (decay) states for which Re(k)>0, Im(k)<0,

d) incoming (capture) states for which Re(k)<0, Im(k)<0.

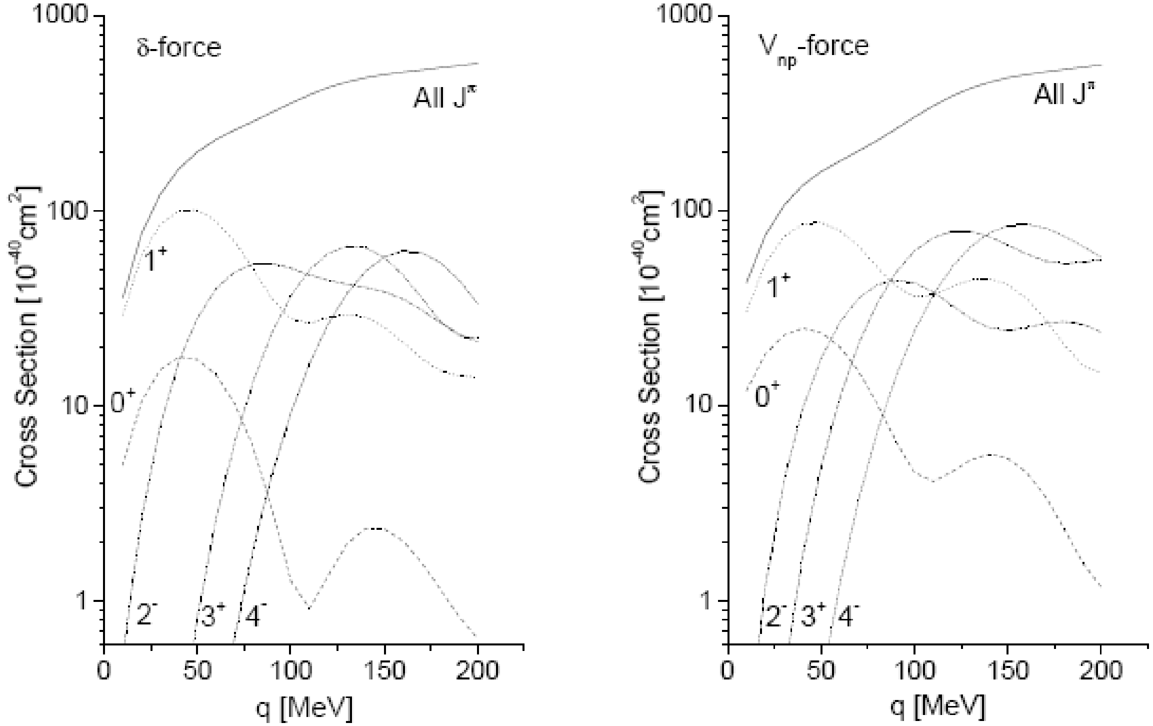
Proton states after the diagonalization of the Woods Saxon plus Coulomb potential

lj	E (real) [MeV]	E (imag) [MeV]	
$h_{9/2}$	-3.784	0	bound
$f_{7/2}$	-3.541	0	bound
$i_{13/2}$	-1.844	0	bound
$p_{3/2}$	-0.690	0	bound
$f_{5/2}$	-0.518	0	bound
$p_{1/2}$	0.491	0	quasi-bound
$g_{9/2}$	4.028	0	quasi-bound
$i_{11/2}$	5.434	0	quasi-bound
$j_{15/2}$	5.960	0	quasi-bound
$d_{5/2}$	6.748	-0.002	resonant
$s_{1/2}$	7.843	-0.037	resonant
$g_{7/2}$	8.087	-0.001	resonant
$d_{3/2}$	8.530	-0.028	resonant
$f_{7/2}$	12.748	-0.652	resonant
$h_{11/2}$	11.390	-0.022	resonant
$k_{17/2}$	14.066	-0.001	resonant
$h_{9/2}$	15.964	-0.393	resonant
$j_{13/2}$	15.086	-0.005	resonant
$i_{13/2}$	18.143	-0.575	resonant

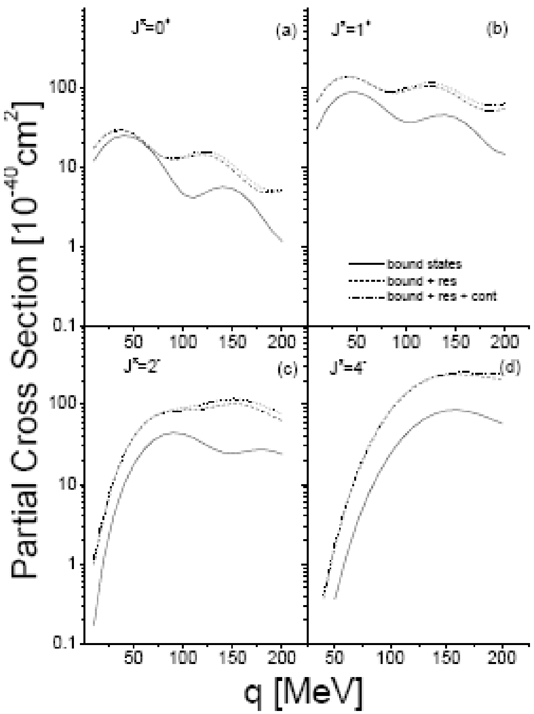


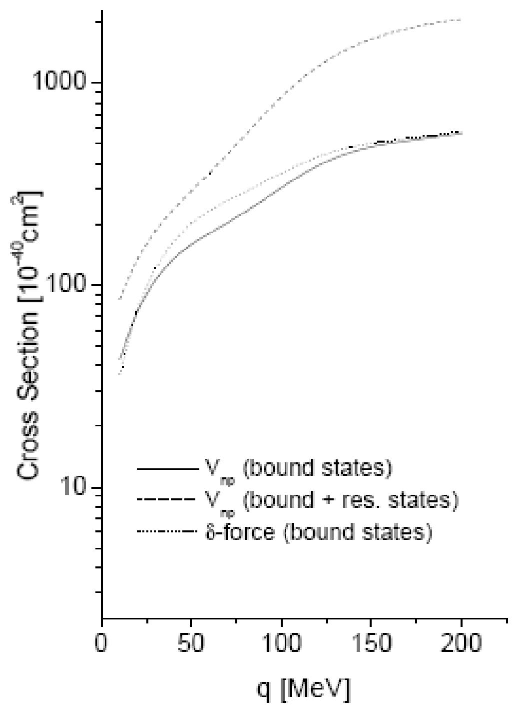
2
0
-2
r Ψ [fm$^{-1/2}$]
Imaginary
Real
0 2 4 6 8 10 12 14 16
Radius r [fm]

Dependence of the results on the chosen residual proton-neutron interaction



Effects due to the inclusion of resonant and continuum states





The results of the calculations, show that:

1. the largest contributions to the considered channels of the cross section are given by nuclear excitations where bound and resonant states participate as proton single-particle states, and
2. the contribution of single-particle states in the continuum is, for all practical purposes, negligible.

Some final words after these examples....

Acknowledgements

I want to express my gratitude to our colleagues, who have contributed to the development of the field and from whom I have had the privilege of learning the techniques presented in these lectures.

Since these is a very long list, indeed, I warmly and gratefully thanks all of them in the persons of Profs. A. Bohm, M. Gadella, and R.J. Liotta

Within few months three of the most outstanding physicist of the twenty century: *Aage* Bohr (NBI), Carlos Guido Bollini (UNLP) and Marcos Moshinsky (UNAM) passed away.

I have had the privilege of knowing them. They had in common the joy of thinking physics and the generosity of sharing their feelings with their students and colleagues .

I hope that their trajectories and
memories will remain with us and with the younger generations of physicists in the years to come.

Thanks are due, with pleasure, to the organizers of this ELAF and thanks to all of you.

Pues nada!!!!