# The Dirac (Moshinsky) Oscillator: Theory and applications

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"As mentioned in the Introduction, we have presented in this volume mainly applications of the harmonic oscillator related to our own work or to that of those with which we have come in contact...
...A complete analysis of the subject would require an encyclopedia, within which one of the volumes could be the present book."

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- Review
  - Why a linear equation in phase space?
  - Solutions: $2 \times 2$ system of equations
  - Spectrum, wavefunctions
  - Non-relativistic limit
- Lorentz invariance, Pauli coupling
- Other potentials (factorization method)

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  - Finite vs. infinite degeneracies in $d = 1, 2, 3$
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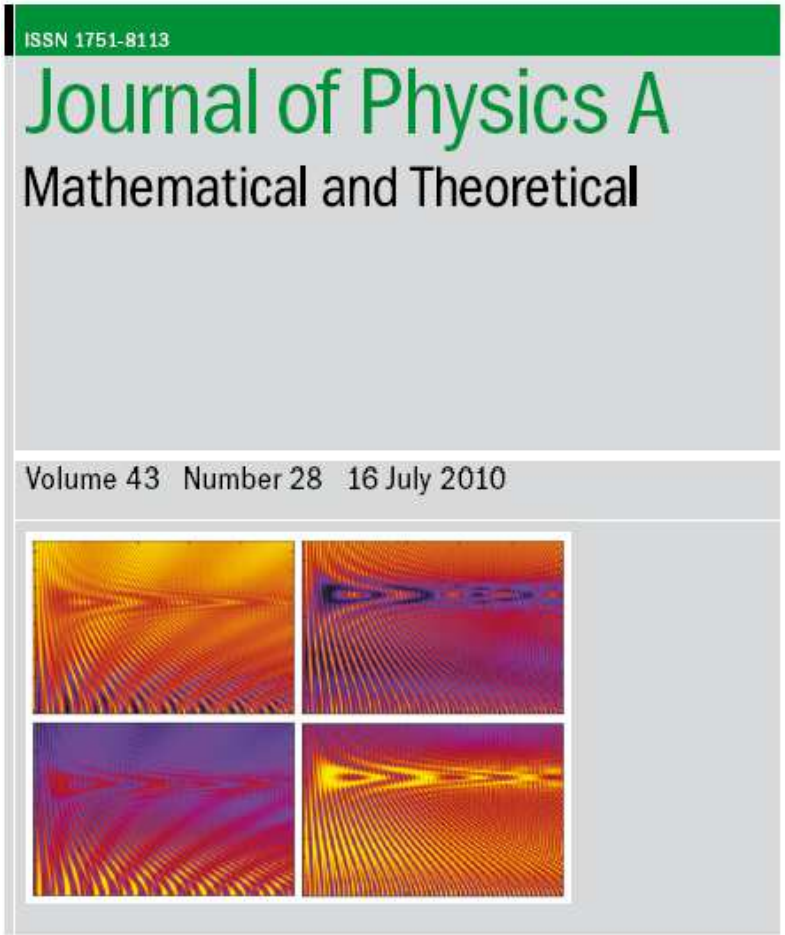
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# Motivation

The harmonic oscillator is the paradigm of integrability and solvability with applications to many branches of physics. Is it possible to promote all these features to a relativistic quantum-mechanical model? The obvious way to proceed is to add a harmonic oscillator potential to the stationary Klein-Gordon operator. However, the Dirac equation requires the "square root" of such operators. For this reason $x, p$ should appear linearly in a Dirac operator which parallels the phase-space symmetry of the usual oscillator.

# Motivation

- Our purpose is to review the construction of an interaction for relativistic systems (particles) producing bound states for arbitrarily high energies with analytically solvable spectrum. Lorentz invariance is crucial.
- This was achieved by Moshinsky and Szczepaniak (1989) with further generalizations to describe interacting particles (1994) through Poincare invariant equations.

# The Klein-Gordon oscillator

A naive approach to the problem is to propose a one-particle relativistic equation in the form

$$(c^2\hbar^2\triangle + m^2c^4 + \frac{1}{2}\omega^2 r^2)\phi = 0 \tag{1}$$

with the trivial result that energies become $(\hbar = 1 = c)$

$$E^2 - m^2 = 2\omega(n + \frac{3}{2}) \tag{2}$$

However, Lorentz invariance is not clear from the outset. It is also necessary to find a first order equation in time for a good application to hamiltonian systems in quantum mechanics.

# The Dirac oscillator

Moshinsky and Szczepaniak introduced a hamiltonian of the form

$$H = c\boldsymbol{\alpha} \cdot (\mathbf{p} \pm i\omega m\beta\mathbf{r}) + mc^2\beta \tag{3}$$

Their purpose was to generalize the symmetry of the harmonic oscillator to the context of relativistic wave equations. Both coordinate and momentum operators must appear in linear form in order to preserve integrability. The symmetry group includes now the Dirac algebra and decomposes naturally into $O(4)$ (compact component representing an oscillator) and $O(3,1)$ (non-compact component representing states with infinite degeneracy)

# The Dirac oscillator

Here we deal with the Dirac equation with a non-minimal coupling which is linear in coordinates. Lorentz invariant wave equation reads

$$\left(\gamma^{\mu}\left[p_{\mu}-i\omega r_{\perp\mu}u_{\nu}\gamma^{\nu}\right]+1\right)\Psi=0 \tag{4}$$

where $\gamma^{\mu}$ are Dirac matrices and

$$r_{\perp\mu}=r_{\mu}-(r^{\nu}u_{\nu})u_{\mu} \tag{5}$$

the $u_{\nu}$ being a time-like four vector such that $(u_{\nu})=(1,0,0,0)$ for some inertial frame. There, (4) can be written as

$$H\Psi=i\frac{\partial\Psi}{\partial t} \tag{6}$$

# The Dirac oscillator

$$H = \boldsymbol{\alpha} \cdot (\mathbf{p} - i\omega\beta\mathbf{r}) + \beta \tag{7}$$

with $\beta = \gamma^0$, $\alpha^i = \beta\gamma^i$, $i = 1, 2, 3$. Stationary form

$$H\Psi = E\Psi \tag{8}$$

with solutions

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{9}$$

satisfying

$$\left(p^2 + \omega^2 r^2 + 1 - 3\omega - 4\omega\mathbf{L}\cdot\mathbf{S}\right)\psi_1 = E^2\psi_1 \tag{10}$$

$$\left(p^2 + \omega^2 r^2 + 1 + 3\omega + 4\omega\mathbf{L}\cdot\mathbf{S}\right)\psi_2 = E^2\psi_2 \tag{11}$$

# The Dirac oscillator

$$\psi_1 = A_{Njl}|N(l,\tfrac{1}{2})jm\rangle \tag{12}$$

$$\psi_2 = (E+1)^{-1}\mathbf{S}\cdot(\mathbf{p}-i\omega\mathbf{r})\psi_1 \tag{13}$$

Energies given by

$$E^2_{Njl} = 1+\omega \begin{array}{ll} 2(N-j)+1 & l=j-\frac{1}{2} \\ 2(N+j)+3 & l=j+\frac{1}{2} \end{array} \tag{14}$$

$$\Psi_\pm = \begin{pmatrix} \psi_1^\pm \\ \psi_2^\pm \end{pmatrix}, \quad \text{if} \quad \pm E > 0 \tag{15}$$

The completeness of these eigenfunctions has been proved elsewhere.

# The Dirac oscillator

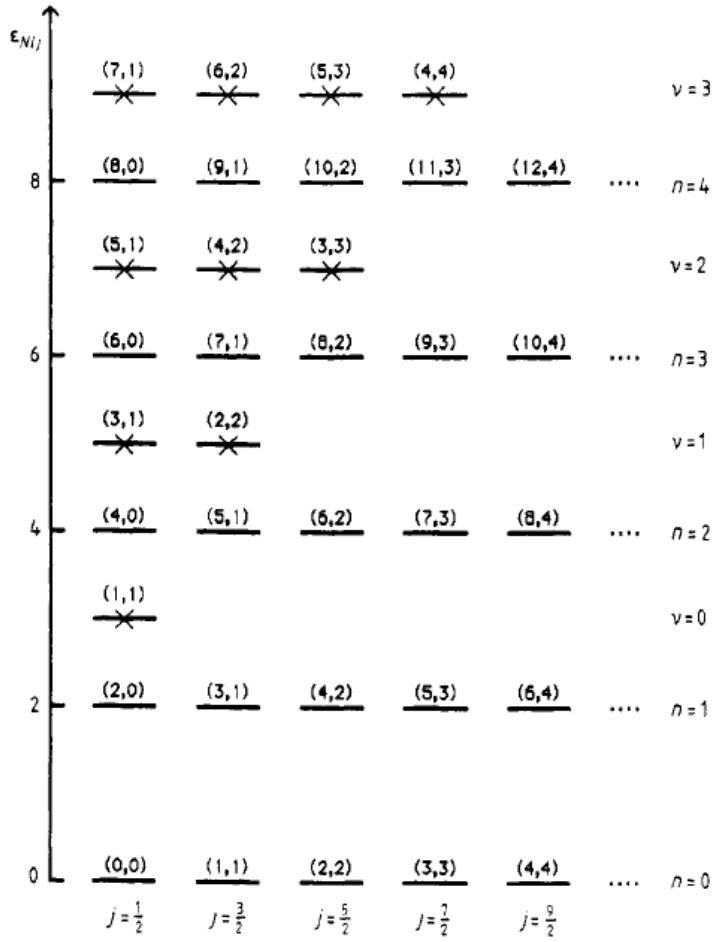


**Figure 1.** Energy spectrum of the Dirac oscillator. The abscissa corresponds to the total angular momentum and the ordinate to $\epsilon_{Nlj}$. The levels are labelled by $(N, l)$. Those belonging to $\mathcal{H}^{(+)}$ are marked by a cross, while those belonging to $\mathcal{H}^{(-)}$ are unmarked. The corresponding values of $\nu$ or $n$ are indicated in the right column.

# Non-relativistic limit

Restoring the units

$$(E^2 - m^2c^4)\psi_1 = \left(c^2(p^2 + \omega^2 m^2 r^2) - 3\hbar\omega mc^2 - 4\frac{\omega}{\hbar}mc^2\mathbf{L}\cdot\mathbf{S}\right)\psi_1 \tag{16}$$

one has $\epsilon = E - mc^2 \ll mc^2$, leading to

$$\epsilon\psi_1 = \left(H_{HO} - \frac{3}{2}\hbar\omega - 2\frac{\omega}{\hbar}\mathbf{L}\cdot\mathbf{S}\right)\psi_1 \tag{17}$$

The infinite degeneracy does not disappear, but the negative energy solutions decouple from small components as expected

# The covariant equation

For convenience, let us eliminate the frequency from our units and leave the rest mass. The equation leading to the Dirac oscillator hamiltonian is

$$[\gamma_\mu(p^\mu + i\gamma_\nu u^\nu r_\perp^\mu) + m]\psi = 0, \tag{18}$$

where $u_\nu$ is unit time-like vector which defines an inertial observer. The perpendicular projection of coordinates is $r_\perp^\mu = r^\mu - (r^\nu u_\nu)u^\mu$ and

$$\gamma_j = \begin{pmatrix} 0 & i\sigma_j \\ i\sigma_j & 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}. \tag{19}$$

# Pauli coupling

The Dirac eq. can be written also as

$$[\gamma_\mu p^\mu + m + S_{\mu\nu}F^{\mu\nu}]\psi = 0 \tag{20}$$

with the choice $F^{\mu\nu} = u^\mu r^\nu - u^\nu r^\mu$. The meaning of the external field can be found by noting that

$$\partial_\mu F^{\mu\nu} = -u^\nu, \tag{21}$$

i.e. the vector $u^\nu$ can be interpreted as a current. In the frame of reference $(1, 0, 0, 0)$ we have a uniform charge density filling the space.

# Solvable Extensions

A supersymmetric formulation (Castaños et al.)

$$[Q_a, Q_b]_+ = \delta_{ab}(H^2 - 1), \quad [Q_a, H^2] = 0 \tag{22}$$

$$Q_1 = \begin{pmatrix} 0 & \boldsymbol{\sigma} \cdot \mathbf{A}^\dagger \\ \boldsymbol{\sigma} \cdot \mathbf{A} & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -i\boldsymbol{\sigma} \cdot \mathbf{A}^\dagger \\ i\boldsymbol{\sigma} \cdot \mathbf{A} & 0 \end{pmatrix}. \tag{23}$$

reveals that other choices allow solvability: $\mathbf{A} = \mathbf{p} + iG(r)\mathbf{r}$, with $G(r)$ a function leading to H.O. or Coulomb problems with centrifugal barriers. We shall use an alternative notation to understand infinite degeneracies in connection with dimensionality.

# Hilbert space

The Lorentz group is locally isomorphic to $SU(2) \times SU^*(2)$. The Hilbert space is $L_2(C) \times SU(2) \times SU^*(2)$

$$H = \boldsymbol{\alpha} \cdot (\mathbf{p} + i\beta\mathbf{r}) + m\beta \tag{24}$$

We shall use a representation of the Dirac matrices given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}. \tag{25}$$

Quantum optical representation (why?)

# Hilbert space

With this notation we may introduce the concept of $*-$spin through the vector $\Sigma_i$, whose projection eigenvalues account for big and small components of spinors. Upon rotations, this projection also gives solutions with positive and negative energies.

$$\Sigma_+ = \begin{pmatrix} 0 & \mathbf{1}_2 \\ 0 & 0 \end{pmatrix} = \sigma_+ \otimes \mathbf{1}_2, \qquad \Sigma_- = (\Sigma_+)^\dagger, \qquad \Sigma_3 = \beta \tag{26}$$

The Hamiltonian can be written in algebraic form

$$H = \Sigma_+ \mathbf{S}\cdot\mathbf{a} + \Sigma_- \mathbf{S}\cdot\mathbf{a}^\dagger + m\Sigma_3, \tag{27}$$

# Algebraic structure

The dependence of $H$ on ladder operators shows the invariants $I = \mathbf{a}^\dagger \cdot \mathbf{a} + \frac{1}{2}\Sigma_3$, $I' = (\mathbf{a}\cdot\boldsymbol{\sigma})^\dagger(\mathbf{a}\cdot\boldsymbol{\sigma}) + \frac{1}{2}\Sigma_3$. A pair of states with angular momentum $j$ and such that $I|\quad\rangle = (2n + j - 1)|\quad\rangle$ is given by

$$|\phi_1\rangle = |n, (j - 1/2, 1/2)j, m_j\rangle|-\rangle, \quad |\phi_2\rangle = |n - 1, (j + 1/2, 1/2)j, m_j\rangle|+\rangle. \quad (28)$$

Another pair of states with the same angular momentum $j$ but with $I|\quad\rangle = (2n + j)|\quad\rangle$ is

$$|\phi_3\rangle = |n, (j + 1/2, 1/2)j, m_j\rangle|-\rangle, \quad |\phi_4\rangle = |n - 1, (j - 1/2, 1/2)j, m_j\rangle|+\rangle. \quad (29)$$

# Algebraic structure

The $2 \times 2$ blocks of $H$ obtained from these states can be evaluated.

$$H(j, 2n + j - 1) = \begin{pmatrix} -m & \sqrt{2n} \\ \sqrt{2n} & m \end{pmatrix}, \tag{30}$$

$$H(j, 2n + j) = \begin{pmatrix} -m & \sqrt{2(n+j)} \\ \sqrt{2(n+j)} & m \end{pmatrix} \tag{31}$$

leading to the well known energies $E^2 = m^2 + 2(n + j)$ and $E^2 = m^2 + 2n$. Infinite and finite degeneracies come from these two blocks respectively.

# Algebraic structure

The discussion on the algebraic structure above can be implemented directly in 1 and 2 spatial dimensions.

$$A_R = a_1 + ia_2, \qquad A_L = a_1 - ia_2 = (A_R)^* \tag{32}$$

with the properties

$$[A_R, A_L] = [A_R, (A_L)^*] = 0, \quad [A_R, A_R^\dagger] = [A_L, A_L^\dagger] = 4. \tag{33}$$

# Algebraic structure

The low dimensional hamiltonians are

$$H^{(1)} = \alpha_1 \left(p + i\beta x\right) + m\beta, \tag{34}$$

with $\alpha_1 = -\sigma_1, \beta = \sigma_3$ and

$$H^{(2)} = \sum_{i=1,2} \alpha_i (p_i + i\beta r_i) + m\beta, \tag{35}$$

with $\alpha_1 = -\sigma_2, \alpha_2 = -\sigma_1, \beta = \sigma_3$.

# Algebraic structure

These hamiltonians can be cast in algebraic form as

$$H^{(1)} = \sigma_+ a + \sigma_- a^\dagger + m\sigma_3 \tag{36}$$

$$H^{(2)} = \sigma_+ A_R + \sigma_- A_R^\dagger + m\sigma_3 \tag{37}$$

Both of them have a $2 \times 2$ structure: The spin is absent in one spatial dimension and $\sigma_\pm$ corresponds to $*-$spin, while in two dimensions $\sigma_3$ generates the $U(1)$ spin.

# Algebraic structure

The solvability can be viewed again as a consequence of the invariants

$$I^{(1)} = a^\dagger a + \frac{1}{2}\sigma_3 \tag{38}$$

$$I^{(2)} = A_R A_R^\dagger + \frac{1}{2}\sigma_3, \qquad J_3 = A_R A_R^\dagger - A_L A_L^\dagger + \frac{1}{2}\sigma_3 \tag{39}$$

The two dimensional case exhibits some peculiarities. The conservation of angular momentum $J_3$ comes from the combination of $\boldsymbol{\sigma}$ and $A_R$ in $H^{(2)}$, together with the absence of $A_L, A_L^\dagger$. This absence is also responsible for the infinite degeneracy of all levels. On the other hand, the three dimensional example is manifestly invariant under rotations due to its dependence on $\mathbf{S}\cdot\mathbf{a}$ and $\mathbf{S}\cdot\mathbf{a}^\dagger$ and its infinite degeneracy comes from the infinitely degenerate operator $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{a})^\dagger$.

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# The many body Dirac equation

- The success of Moshinsky's work related to the harmonic oscillator of arbitrary particles and dimensions is due to the fact that the results provided a good basis to solve variational problems in bound composite systems
- The idea is to extend this success to relativistic quantum mechanics. However, we need a model which allows the integrability and solvability we are seeking for.

# The Foldy-Wouthuysen transformation

There exists a unitary operator which transforms the Dirac hamiltonian into a diagonal operator in spinorial components. In other words, it finds the basis in which the z component of $*$-spin gives the positive and negative energies of the system.
Such transformation can be carried out explicitly for exactly solvable problems such as the free particle and the Dirac oscillator. The idea is to express the hamiltonian in terms of its even and odd parts and find $S$ such that

$$H_{FW} = e^{iS} H_D e^{-iS} = \text{even} \tag{1}$$

In our example, $iS = \beta(\alpha \cdot \pi)\theta$, $\tan(2\theta\alpha \cdot \pi) = \alpha \cdot \pi$

# Relativistic Composite Particles

## Poincaré invariance of the many body problem

We propose a generalization of the Dirac equation for a system of many particles. It is defined such that, in the frame of reference where the center of mass is at rest, we recover a hamiltonian of the form

$$H = \sum_{i}^{N} H_i + V(\mathbf{x}_1, ..., \mathbf{x}_N) \tag{2}$$

where $H_i$ is the Dirac hamiltonian of the $i$-th particle. The potential $V$ is assumed to be independent of the center of mass. Such an equation is

$$\left[ \sum_{s=1}^{N} \Gamma_s (\gamma_s^\mu p_{\mu s} + m_s + \Gamma_s V(x_\perp^s)) \right] \psi = 0 \tag{3}$$

The relative coordinates and the time-like relative coordinates are given respectively by

$$x^{st}_{\mu} = x^{s}_{\mu} - x^{t}_{\mu}, \quad x^{st}_{\perp\mu} = x^{st}_{\mu} - x^{st}_{\tau} u^{\tau} u_{\mu}, \tag{4}$$

We use the time-like unit vector in the form

$$u_{\mu} = (-P_{\tau} P^{\tau})^{-1/2} P_{\mu}. \tag{5}$$

For convenience we have defined

$$\Gamma = \prod_{r=1}^{N} \gamma_r^\mu u_\mu, \qquad \Gamma_s = (\gamma_s^\mu u_\mu)^{-1}\Gamma. \tag{6}$$

Taking $P^i = 0$ and $H = P^0$ in (3), one recovers (2).

# The cockroach nest

For commuting Dirac hamiltonians one expects that the total FW transformation can be decomposed into individual factors corresponding to each hamiltonian. We shall define the multiparticle FW transformation in the next slides, but let us note that for free particles we should obtain

$$H_{FW} = \sum_{i=1}^{N} \beta_i \sqrt{p_i^2 + m_i^2} \tag{7}$$

where it becomes evident that the energies are now added with 'wrong' signs due to the $\beta$ matrices. This means that the transformation to even hamiltonians contains both particle and anti-particle solutions without a correction of the signs in front of their kinetic energies. One has to project the final result onto the purely positive component, otherwise we would obtain an extraordinary infinite degeneracy.

## The many body Foldy-Wouthuysen transformation

With the aim of characterizing the spectrum of a multibody system with interactions, we seek for an expansion of $H$ in terms of inverse powers of the rest mass. Such an expasion should allow the identification of positive and negative energies of the model. For one particle in a potential $V$, we have

$$H = \mathcal{O} + \mathcal{E} + V, \quad \mathcal{O} = \boldsymbol{\alpha} \cdot \mathbf{p}, \quad \mathcal{E} = m\beta \tag{8}$$

we apply a unitary operator $U = \exp(iS)\exp(iS')\exp(iS'')$,

$$S = \frac{-i\beta}{2m}\mathcal{O}, \quad S' = \frac{-i\beta}{2m}\mathcal{O}', \quad S'' = \frac{-i\beta}{2m}\mathcal{O}''$$

$$\mathcal{O}' = \frac{\beta}{2m}[\boldsymbol{\alpha} \cdot \mathbf{p}, V], \quad \mathcal{O}'' = \frac{-(\boldsymbol{\alpha} \cdot \mathbf{p})p^2}{3m^2}, \quad H' = UHU^\dagger \tag{9}$$

Expanding up to $1/(mass)^3$ in the kinetic energy, $1/(mass)^2$ in the potential, we have

$$H' = \hat{H} + V,\ \hat{H} = \beta\left(m + \frac{p^2}{2m} - \frac{p^4}{8m^3}\right)$$
$$+\frac{1}{4m^2}\mathbf{s}\cdot\Big[(\mathbf{p}\times\mathbf{E}) - (\mathbf{E}\times\mathbf{p})\Big] + \frac{1}{8m^2}\nabla^2 V \tag{10}$$

with $\mathbf{E} = -\nabla V$, $\mathbf{S} = \frac{-i}{4}\boldsymbol{\alpha}\times\boldsymbol{\alpha}$. For two particles $H = H_1 + H_2 + V(\mathbf{r}_1, \mathbf{r}_2)$. Applying successively $U_1 = \exp(iS_1)$ and $U_2 = \exp(iS_2)$ one gets

$$U_2U_1H(U_2U_1)^\dagger = \hat{H}_1 + \hat{H}_2 + V + \text{higher} \quad \text{order} \tag{11}$$

In the general case with $n$ particles, one has

$$H = \sum_{i=1}^{N} H_i + V(\mathbf{r}_1, ..., \mathbf{r}_N) \tag{12}$$

$$H' = U_N ... U_1 H (U_N ... U_1)^\dagger = \sum_{i=1}^{N} \hat{H}_i + V(\mathbf{r}_1, ..., \mathbf{r}_N) \tag{13}$$

with

$$\hat{H}_t = \beta_t \left( m_t + \frac{p_t^2}{2m_t} - \frac{p_t^4}{8m_t^3} \right) + \frac{1}{4m_t^2} \mathbf{s}_t \cdot (\mathbf{p}_t \times \mathbf{E}_t - \mathbf{E}_t \times \mathbf{p}_t) + \frac{1}{8m_t^2} \nabla_t^2 V,$$
$$t = 1, 2, \cdots n \tag{14}$$

## Application to a two quark system

The hamiltonian is

$$H' = (\beta_1 + \beta_2)\left(m + \frac{p^2}{2m} - \frac{p^4}{8m^3}\right) + V$$
$$+\frac{1}{4m^2}\left(\mathbf{s}_1 + \mathbf{s}_2\right) \cdot \left[(\mathbf{p} \times \mathbf{E}) - (\mathbf{E} \times \mathbf{p})\right] + \frac{1}{4m^2}\nabla^2 V \qquad (15)$$

with the potential $V = \frac{1}{2}m\omega^2(\mathbf{r}_1 - \mathbf{r}_2)^2$.

In the center of mass frame, the choice of positive energy components reduces the hamiltonian to

$$H' = \left(2m + 3\frac{\omega^2}{8m}\right) + \left(\frac{p^2}{m} + \frac{m\omega^2 r^2}{4} + \frac{\omega^2}{4m}\mathbf{S} \cdot \mathbf{L}\right) - \frac{p^4}{4m^3} \qquad (16)$$

with $\mathbf{r}, \mathbf{p}$ relative coordinate and momentum. The spectrum of the problem is found by diagonalizing the matrix with elements given by

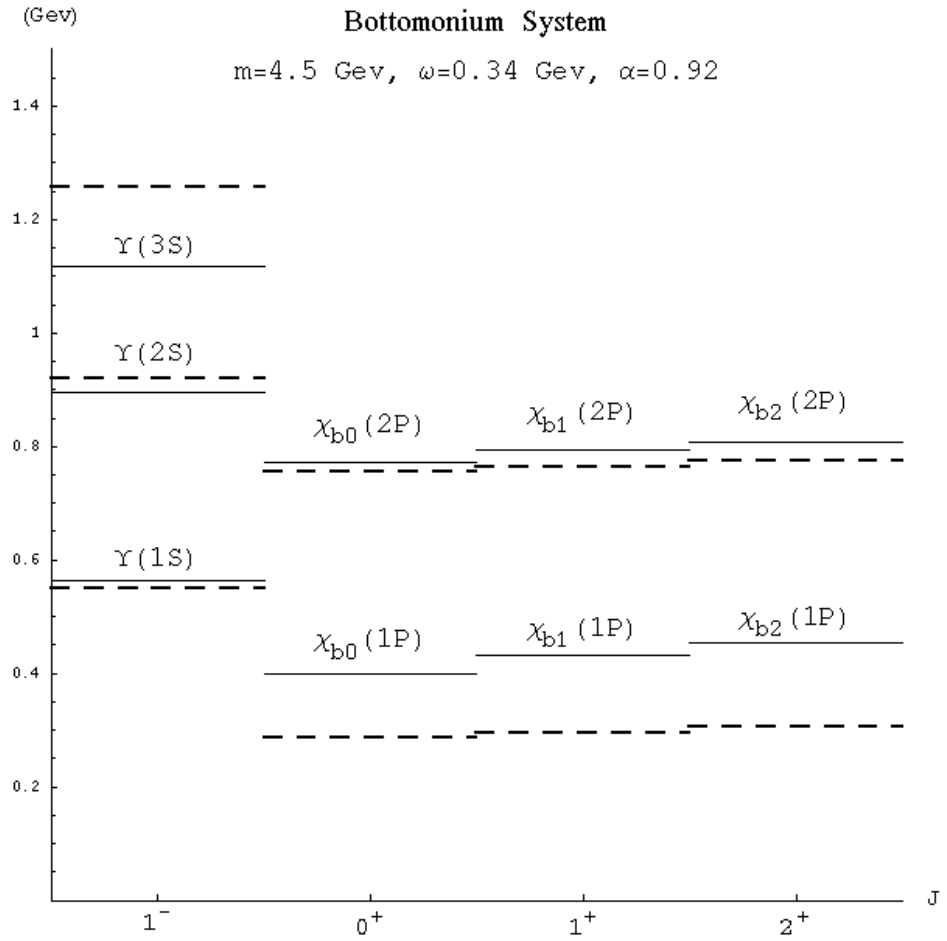
$$< n'l, \left(\tfrac{1}{2}\tfrac{1}{2}\right)S; j, m|H'|nl, \left(\tfrac{1}{2}\tfrac{1}{2}\right)S, j, m >$$
$$= \left(2m + \tfrac{3\omega^2}{8m} + \omega\left(2n + l + \tfrac{3}{2}\right) + \tfrac{\omega^2}{8m}[j(j+1) - l(l+1) - s(s+1)]\right)\delta_{nn'}$$
$$-\tfrac{1}{4m^3} < n'l'|p^4|nl > \qquad (17)$$

where we use two particle harmonic oscillator states with spin, *i.e.*

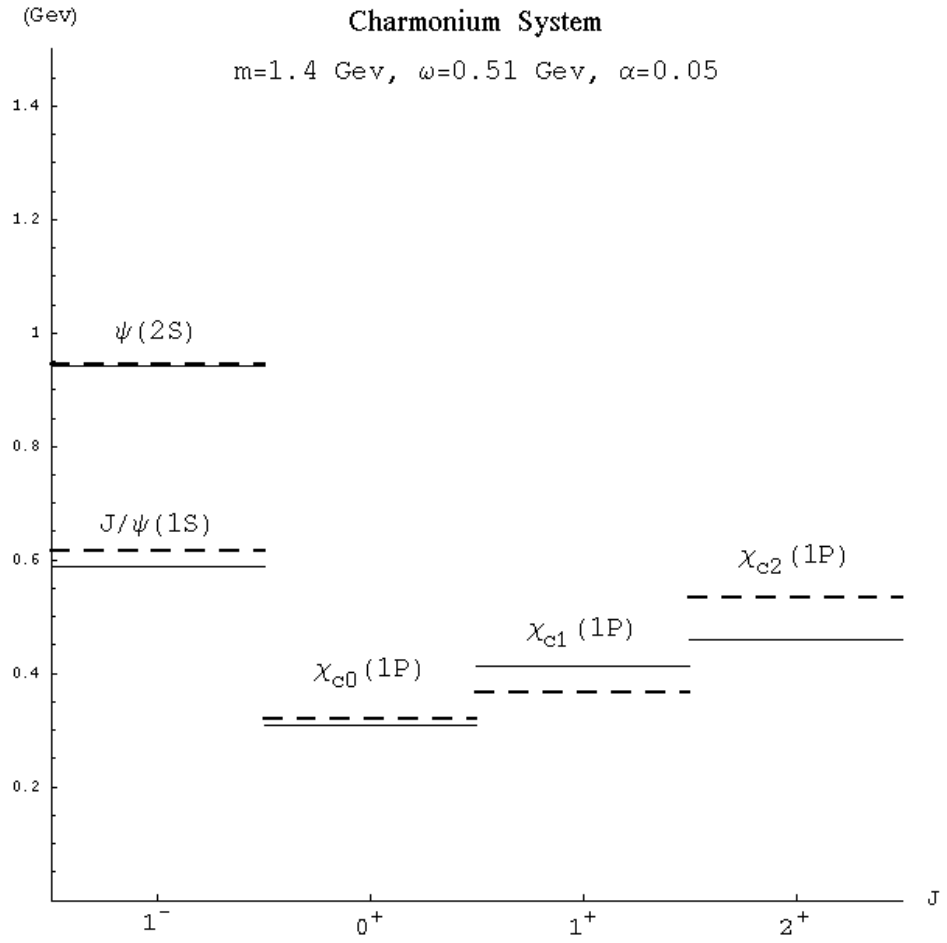
$$|nl, \left(\frac{1}{2}\frac{1}{2}\right)S; j, m > \equiv \sum_{\mu,\sigma} < l\mu, S\sigma|jm > |nl\mu > |\left(\frac{1}{2}\frac{1}{2}\right)S\sigma > \qquad (18)$$

We take $N \leq N_{max}$ to get a finite matrix.

As an application, one can describe the mass spectrum of binary systems such as bottomonium or charmonium. It is possible to introduce quartic corrections to the potential in order to obtain more realistic spectra $V' = -\frac{am\omega^4 r^4}{16}$. The FW transformation of such a term yields next order corrections, therefore we neglect them. The coupling constants and the rest mass are taken as adjustable parameters. They are fitted to experimental data using least dispersion.




Energy comparison. Solid: experimental. Dotted: theory.




Energy comparison. Solid: experimental. Dotted: theory.

## Application to the three body problem

The hamiltonian is now

$$H' = \sum_{t=1}^{3} \beta_t \left( m_t + \frac{p_t^2}{2m_t} - \frac{p_t^4}{8m_t^3} \right) + \frac{1}{4m_t^2} \mathbf{s}_t \cdot (\mathbf{p}_t \times \mathbf{E}_t - \mathbf{E}_t \times \mathbf{p}_t) + \frac{1}{8m_t^2} \nabla_t^2 V + V \tag{19}$$

with a potential

$$V = \frac{M\omega^2}{6} \left[ (\mathbf{r}_1 - \mathbf{r}_2)^2 + (\mathbf{r}_2 - \mathbf{r}_3)^2 + (\mathbf{r}_3 - \mathbf{r}_1)^2 \right] \tag{20}$$

# Jacobi coordinates

The harmonic oscillator for $n$ particles

$$H = \tfrac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{\omega^2}{2n}\sum_{i,j=1}^{n}(\mathbf{r}_i - \mathbf{r}_j)^2 \tag{21}$$

can be decoupled into $n-1$ oscillators by using the Jacobi coordinates

$$\begin{aligned}(\dot{p}_s)_j &= [s(s+1)]^{-1/2}\sum_{t=1}^{s}\left((p_t)_j - (p_{s+1})_j\right), s = 1,\ldots,n-1,\\ (\dot{p}_n)_j &= n^{-1/2}\sum_{t=1}^{n}(p_t)_j\end{aligned} \tag{22}$$

In Jacobi coordinates without the center of mass we have

$$
\begin{aligned}
H' &= Mc^2 + \left(\frac{1}{4}\dot{p}_1^2 + \frac{1}{12}\dot{p}_2^2\right)\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \frac{1}{3m_3}\dot{p}_2^2 + \frac{1}{\sqrt{12}}\dot{p}_{12}\left(\frac{1}{m_1} - \frac{1}{m_2}\right) \\
&- \frac{1}{8m_1^3c^2}\left(\frac{1}{4}\dot{p}_1^4 + \frac{1}{36}\dot{p}_2^4 + \frac{1}{3}\dot{p}_{12}^2 + \frac{1}{6}\dot{p}_1^2\dot{p}_2^2 + \frac{1}{\sqrt{3}}\dot{p}_{12}\dot{p}_1^2 + \frac{1}{3\sqrt{3}}\dot{p}_{12}\dot{p}_2^2\right) \\
&- \frac{1}{8m_2^3c^2}\left(\frac{1}{4}\dot{p}_1^4 + \frac{1}{36}\dot{p}_2^4 + \frac{1}{3}\dot{p}_{12}^2 + \frac{1}{6}\dot{p}_1^2\dot{p}_2^2 - \frac{1}{\sqrt{3}}\dot{p}_{12}\dot{p}_1^2 - \frac{1}{3\sqrt{3}}\dot{p}_{12}\dot{p}_2^2\right) \\
&+ \frac{1}{18m_3^3c^2}\dot{p}_2^4\frac{M\omega^2}{8c^2}\left[\frac{1}{m_1^2}\mathbf{S}_1 \cdot \left(\dot{\mathbf{L}}_1 + \frac{1}{3}\dot{\mathbf{L}}_2 + \frac{1}{\sqrt{3}}\dot{\mathbf{L}}_{12}\right)\right. \\
&+ \left.\frac{1}{m_2^2}\mathbf{S}_2 \cdot \left(\dot{\mathbf{L}}_1 + \frac{1}{3}\dot{\mathbf{L}}_2 - \frac{1}{\sqrt{3}}\dot{\mathbf{L}}_{12}\right) - \frac{8}{3m_3^2}\mathbf{S}_3 \cdot \dot{\mathbf{L}}_2\right] \\
&+ \frac{M\hbar^2\omega^2}{8c^2}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} + \frac{1}{m_3^2}\right) + V
\end{aligned}
\tag{23}
$$

The spectrum is obtained by diagonalizing

$$< n_1', l_1', n_2', l_2', L'; \left(\frac{1}{2}\frac{1}{2}\right)T' \frac{1}{2} S'; j'm' | H' | n_1, l_1, n_2, l_2, L; \left(\frac{1}{2}\frac{1}{2}\right)T \frac{1}{2} S; jm > \quad (24)$$

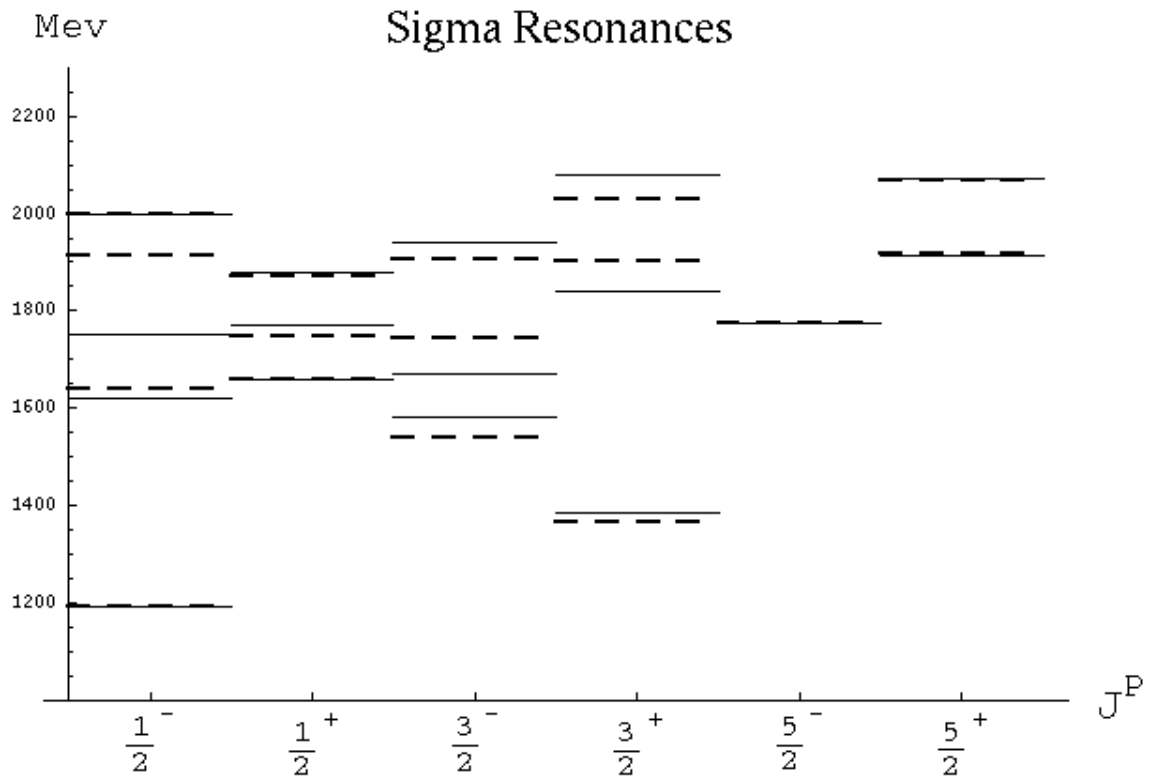
where the states are

$$|n_1, l_1, n_2, l_2, L; \left(\frac{1}{2}\frac{1}{2}\right)T \frac{1}{2} S; jm >=$$

$$\sum_{\mu,\sigma} < L\mu, S\sigma | jm > |n_1, l_1, n_2, l_2, L\mu > | \left(\frac{1}{2}\frac{1}{2}\right)T \frac{1}{2} S\sigma > \quad (25)$$

Matrix elements are computed by means of Racah algebra. We take $N_{max} = 3$.

To achieve a better agreement with experiment, we may introduce a mass an a frequency which depend on the integrals of the motion We include a comparison with the spectra of $\Sigma$ particles (strange baryons). Dotted: Theory. Solid: Experimental



## Table of parameters

| $J^P$ | $\omega$ (en Mev) | M |
|---|---|---|
| $\frac{1}{2}^+$ | 96 | 1.00 |
| $\frac{1}{2}^-$ | 184 | 1.00 |
| $\frac{3}{2}^+$ | 187 | 1.27 |
| $\frac{3}{2}^-$ | 179 | 0.93 |
| $\frac{5}{2}^+$ | 137 | 1.11 |
| $\frac{5}{2}^-$ | 137 | 1.03 |

## Two particle Dirac oscillator

The hamiltonian and the Poincare invariant equation are

$$H = (\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2) \cdot (\mathbf{p} - i\frac{\omega}{2}\mathbf{r}B) + \beta_1 + \beta_2 \tag{26}$$

$$\left[\sum_{s=1,2} \Gamma_s \left(\gamma_s^\mu (p_{\mu s} - i\omega x'_{\perp \mu s}\Gamma) + 1\right)\right] \Psi = 0 \tag{27}$$

The resulting spectrum is given by $E = \pm E_{N,s,j,m}$

$$E_{N,s,j,m} = \begin{array}{lll} 2\sqrt{1+\omega N}, 0 & \text{for} & s = 0, P = (-)^j \\ 2\sqrt{1+\omega(N+2)}, 0 & \text{for} & s = 1, P = (-)^j \\ 2\sqrt{1+\omega(N+1)}, 0 & \text{for} & s = 1, P = -(-)^j \end{array} \tag{28}$$

The wavefunctions are known for all cases indicated before:

$$\Psi = \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{12} \\ \psi_{22} \end{pmatrix}, \quad \begin{pmatrix} \psi_{11} \\ \psi_{22} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a_+ + a_- \\ a_+ - a_- \end{pmatrix} |N(j,0)jm\rangle \tag{29}$$

For $s = 0$. Whenever $s = 1$ and $P = (-)^j$, we have

$$\begin{pmatrix} \psi_{11} \\ \psi_{22} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} b_+ + b_- \\ b_+ - b_- \end{pmatrix} |N(j,1)jm\rangle \tag{30}$$

For $s = 1$, $P = -(-)^j$ the result is

$$\begin{pmatrix} \psi_{11} \\ \psi_{22} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} c_{++} + c_{-+} \\ c_{-+} - c_{++} \end{pmatrix} |N(j+1,1)jm\rangle$$
$$+ \frac{1}{\sqrt{2}} \begin{pmatrix} c_{+-} + c_{--} \\ c_{--} - c_{+-} \end{pmatrix} |N(j-1,1)jm\rangle \qquad (31)$$

where the coefficients $c_{\pm\pm}, a_{\pm}, b_{\pm}$ are determined by the secular equations arising from the Schroedinger equation for the relativistic hamiltonian. Taking into account (26), the stationary equation yields the complementary components of the wavefunction $\psi_{21}$,$\psi_{12}$.

## The three-particle Dirac oscillator

Poincare invariant equation and hamiltonian:

$$\left( n^{-1} \sum_{s=1}^{n} \Gamma_s (\gamma_s^\mu P_\mu) + \sum_{s=1}^{n} \left[ \gamma_s^\mu (p'_{\mu s} - i\omega x'_{\perp \mu s} \Gamma) + 1 \right] \right) \Psi = 0 \qquad (32)$$

$$H\Psi = \sum_{s=1}^{n} \left[ \boldsymbol{\alpha}_s \cdot (\mathbf{p}'_s - i\omega \mathbf{x}'_s B) + \beta_s \right] \Psi = E\Psi \qquad (33)$$

The spectrum is obtained by combining the equations for some of the spinor components of the wavefunction and noting that the total number of quanta of the TWO PARTICLE oscillator is conserved (two Jacobi coordinates). The wavefunctions are

$$\Psi_+ = \begin{pmatrix} \psi_{111} \\ \psi_{122} \\ \psi_{212} \\ \psi_{221} \end{pmatrix}, \quad \Psi_- = \begin{pmatrix} \psi_{112} \\ \psi_{121} \\ \psi_{211} \\ \psi_{222} \end{pmatrix} \tag{34}$$

and they satisfy

$$\mathcal{O}\,\Psi_+ = 0, \quad \mathcal{O} \equiv \mathbf{M} D_-^{-1} \mathbf{M}^\dagger - D_+ \tag{35}$$

with

$$\mathbf{D}_+ = \mathrm{diag}\,(E-3, E+1, E+1, E+1), \tag{36}$$

$$\mathbf{D}_+ = \mathrm{diag}\,(E-1, E-1, E-1, E+3) \tag{37}$$

$$\mathbf{M} = 2i\sqrt{2\omega}\begin{pmatrix} \mathbf{S}_3\cdot\boldsymbol{\eta}'_3 & \mathbf{S}_2\cdot\boldsymbol{\eta}'_2 & \mathbf{S}_1\cdot\boldsymbol{\eta}'_1 & 0 \\ \mathbf{S}_2\cdot\boldsymbol{\eta}'_2 & \mathbf{S}_3\cdot\boldsymbol{\eta}'_3 & 0 & \mathbf{S}_1\cdot\boldsymbol{\eta}'_1 \\ \mathbf{S}_1\cdot\boldsymbol{\eta}'_1 & 0 & \mathbf{S}_3\cdot\eta'_3 & \mathbf{S}_2\cdot\boldsymbol{\eta}'_2 \\ 0 & \mathbf{S}_1\cdot\boldsymbol{\eta}'_1 & \mathbf{S}_2\cdot\boldsymbol{\eta}'_2 & \mathbf{S}_3\cdot\boldsymbol{\eta}'_3 \end{pmatrix}, \tag{38}$$

$$\boldsymbol{\eta}'_s = \boldsymbol{\eta}_s - \frac{1}{3}(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2+\boldsymbol{\eta}_3) \tag{39}$$

$$\boldsymbol{\xi}'_s = \boldsymbol{\eta}'^{\dagger}_s \tag{40}$$


Using the states

$$|n_1, l_1, n_2, l_2(L); \frac{1}{2}\frac{1}{2}(T)\frac{1}{2}(S); JM\rangle =$$

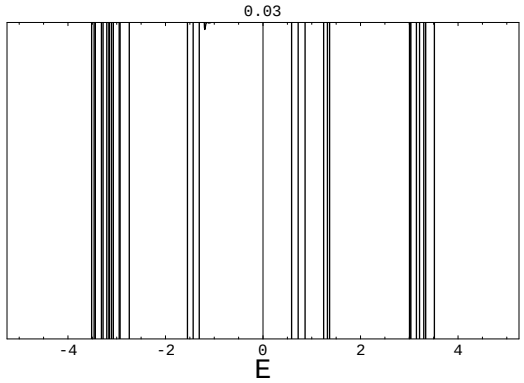
$$\left[[(\dot{r}_1|n_1 l_1) \times (\dot{r}_2|n_2 l_2)]_L \times \left[\left[(1|\frac{1}{2}) \times (2|\frac{1}{2})\right]_T \times (3|\frac{1}{2})\right]_S\right]_{JM} \quad (41)$$

one finds the matrix elements of $\mathcal{O}$. We diagonalize for each number of quanta. The resulting matrices are FINITE. We restrict to $N = 0, 1, 2$. The wavefunctions are finally obtained by finding the null vectors of the matrix $\langle \mathcal{O} \rangle$ for each energy. The complementary components are obtained using the original stationary equation, as before.

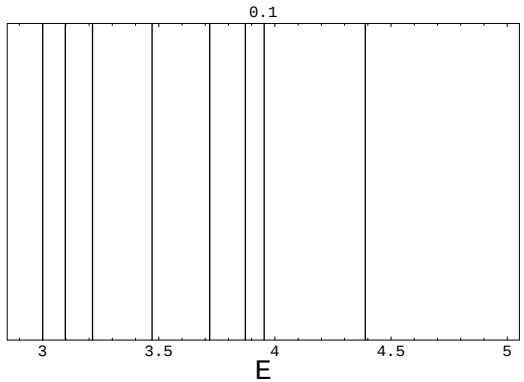
| $N$ | $N_1$ | $N_2$ | $n_1$ | $n_2$ | $l_1$ | $l_2$ | $P$ | $L$ | $J$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | 0 | $S$ |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | - | 1 | $\|1-S\| \leq J \leq 1+S$ |
| 1 | 0 | 1 | 0 | 0 | 0 | 1 | - | 1 | $\|1-S\| \leq J \leq 1+S$ |
| 2 | 2 | 0 | 1 | 0 | 0 | 0 | + | 0 | $S$ |
| 2 | 0 | 2 | 0 | 1 | 0 | 0 | + | 0 | $S$ |
| 2 | 1 | 1 | 0 | 0 | 1 | 1 | + | $0 \leq L \leq 2$ | $\|L-S\| \leq J \leq L+S$ |
| 2 | 2 | 0 | 0 | 0 | 2 | 0 | + | 2 | $\|2-S\| \leq J \leq 2+S$ |
| 2 | 0 | 2 | 0 | 0 | 0 | 2 | + | 2 | $\|2-S\| \leq J \leq 2+S$ |

Table of states for $N_{max} = 2$

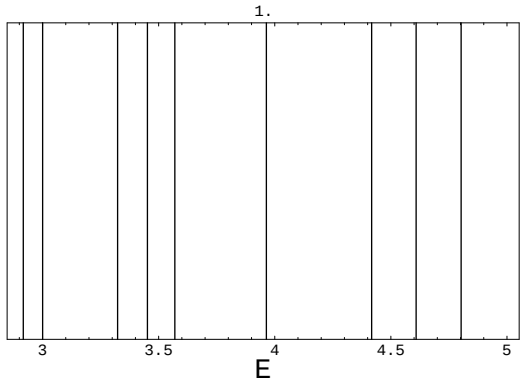
Energies for $\omega = 0.03$. The eigenvalues are distributed in four groups around the values $-3, -1, 1, 3$




Spectrum for $\omega = 0.1$ and $N = 2$,



## Spectrum for $\omega = 1$ and $N = 2$



# One dimensional $n$-particles

The kinetic part of the hamiltonian

$$H = (\mathbf{1} + B) \sum_{i}^{n} \sigma_1^i a_i' + h.c. + \text{mass} \tag{42}$$

is infinitely degenerate. We have removed all other degrees of freedom in order to show that the cockroach nest makes itself present for an arbitrary number of interacting particles. Its elimination is not a trivial task, in despite of our careful choice of observables. To see the infinite degeneracy, apply eigenstates of $\sigma_1^i$ to $(H - \text{mass})^2$. Question: is there a system of one-dimensional Dirac particles which parallels the Calogero model?

# Comment

The original application of the three particle Dirac oscillator contained eigenstates of the permutation group, dealing also with particle statistics. At the end of the calculations, energies and eigenfunctions were obtained and a prediction for the form factor of the proton was given. However, let me quote Moshinsky once again

"We conclude by stressing that we have made a calculation using an harmonic oscillator picture with a single parameter (frequency) and it is as good or as bad as many more complicated ones that start from QCD or that use many more parameters."

# References


- Moshinsky M and Smirnov Y 1996,*The Harmonic Oscillator in Modern Physics* , Hardwood Academic Publishers, Amsterdam.
- Moshinsky M and Szczepaniak A 1989 J Phys A: Math Gen **22** L817.
- Bjorken and Drell, Relativistic Quantum Mechanics
- Moreno M and Zentella A 1989, *J Phys A: Math Gen* **22** L821-LS25.

# The Dirac (Moshinsky) Oscillator: Theory and applications

ELAF July 29, 2010

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- Review
  - Why a linear equation in phase space?
  - Solutions: $2 \times 2$ system of equations
  - Spectrum, wavefunctions
  - Non-relativistic limit
- Lorentz invariance, Pauli coupling
- Other potentials (factorization method)

# Contents. Part I

- An alternative approach for a $d$-dimensional Dirac oscillator
  - The concept of $*$-spin
  - Finite vs. infinite degeneracies in $d = 1, 2, 3$
- Symmetry Lie algebra
- Bibliography

# Contents. Part II

- The many body Dirac equation
  - Poincare invariance
  - Hamiltonian at the center of mass
  - The cockroach nest and extraordinary infinite degeneracy
- The two-particle Dirac oscillator
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- A note on one-dimensional $n$-particle D.O.

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- The Jaynes-Cummings model
  - Identification of observables
  - Realization of a 2-dimensional D.O. in atomic physics
- Extension with non-local external fields
  - Analytical solutions
  - Entanglement
- Non-relativistic electrons in graphene
  - Why a Dirac equation?
  - Deformation of semi-infinite hexagonal lattices

## The three-particle Dirac oscillator

Poincare invariant equation and hamiltonian:

$$\left( n^{-1} \sum_{s=1}^{n} \Gamma_s (\gamma_s^\mu P_\mu) + \sum_{s=1}^{n} \left[ \gamma_s^\mu (p'_{\mu s} - i\omega x'_{\perp \mu s} \Gamma) + 1 \right] \right) \Psi = 0 \qquad (1)$$

$$H\Psi = \sum_{s=1}^{n} \left[ \boldsymbol{\alpha}_s \cdot (\mathbf{p}'_s - i\omega \mathbf{x}'_s B) + \beta_s \right] \Psi = E\Psi \qquad (2)$$

The spectrum is obtained by combining the equations for some of the spinor components of the wavefunction and noting that the total number of quanta of the TWO PARTICLE oscillator is conserved (two Jacobi coordinates). The wavefunctions are

$$\Psi_+ = \begin{pmatrix} \psi_{111} \\ \psi_{122} \\ \psi_{212} \\ \psi_{221} \end{pmatrix}, \quad \Psi_- = \begin{pmatrix} \psi_{112} \\ \psi_{121} \\ \psi_{211} \\ \psi_{222} \end{pmatrix} \tag{3}$$

and they satisfy

$$\mathcal{O}\,\Psi_+ = 0, \quad \mathcal{O} \equiv \mathbf{M} D_-^{-1} \mathbf{M}^\dagger - D_+ \tag{4}$$

with

$$\mathbf{D}_+ = \mathrm{diag}\,(E-3, E+1, E+1, E+1), \tag{5}$$

$$\mathbf{D}_+ = \mathrm{diag}\,(E-1, E-1, E-1, E+3) \tag{6}$$

$$\mathbf{M} = 2i\sqrt{2\omega}\begin{pmatrix} \mathbf{S}_3\cdot\boldsymbol{\eta}'_3 & \mathbf{S}_2\cdot\boldsymbol{\eta}'_2 & \mathbf{S}_1\cdot\boldsymbol{\eta}'_1 & 0 \\ \mathbf{S}_2\cdot\boldsymbol{\eta}'_2 & \mathbf{S}_3\cdot\boldsymbol{\eta}'_3 & 0 & \mathbf{S}_1\cdot\boldsymbol{\eta}'_1 \\ \mathbf{S}_1\cdot\boldsymbol{\eta}'_1 & 0 & \mathbf{S}_3\cdot\eta'_3 & \mathbf{S}_2\cdot\boldsymbol{\eta}'_2 \\ 0 & \mathbf{S}_1\cdot\boldsymbol{\eta}'_1 & \mathbf{S}_2\cdot\boldsymbol{\eta}'_2 & \mathbf{S}_3\cdot\boldsymbol{\eta}'_3 \end{pmatrix}, \tag{7}$$

$$\boldsymbol{\eta}'_s = \boldsymbol{\eta}_s - \frac{1}{3}(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2+\boldsymbol{\eta}_3) \tag{8}$$

$$\boldsymbol{\xi}'_s = \boldsymbol{\eta}'^{\dagger}_s \tag{9}$$


Using the states

$$|n_1, l_1, n_2, l_2(L); \frac{1}{2}\frac{1}{2}(T)\frac{1}{2}(S); JM\rangle =$$

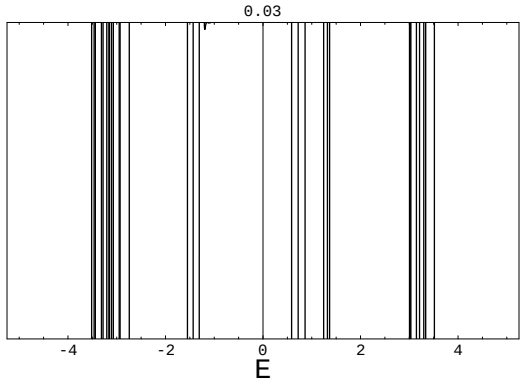
$$\left[[(\dot{r}_1|n_1 l_1) \times (\dot{r}_2|n_2 l_2)]_L \times \left[\left[(1|\frac{1}{2}) \times (2|\frac{1}{2})\right]_T \times (3|\frac{1}{2})\right]_S\right]_{JM} \quad (10)$$

one finds the matrix elements of $\mathcal{O}$. We diagonalize for each number of quanta. The resulting matrices are FINITE. We restrict to $N = 0, 1, 2$. The wavefunctions are finally obtained by finding the null vectors of the matrix $\langle \mathcal{O} \rangle$ for each energy. The complementary components are obtained using the original stationary equation, as before.

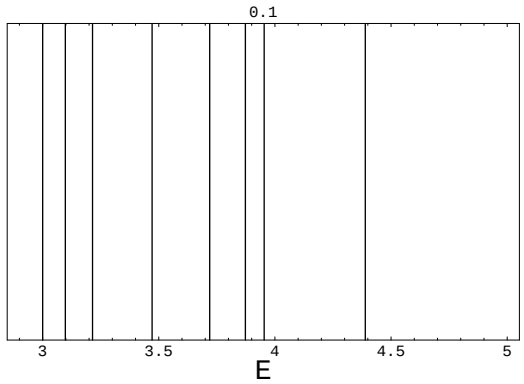
| $N$ | $N_1$ | $N_2$ | $n_1$ | $n_2$ | $l_1$ | $l_2$ | $P$ | $L$ | $J$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | 0 | $S$ |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | - | 1 | $\|1-S\| \leq J \leq 1+S$ |
| 1 | 0 | 1 | 0 | 0 | 0 | 1 | - | 1 | $\|1-S\| \leq J \leq 1+S$ |
| 2 | 2 | 0 | 1 | 0 | 0 | 0 | + | 0 | $S$ |
| 2 | 0 | 2 | 0 | 1 | 0 | 0 | + | 0 | $S$ |
| 2 | 1 | 1 | 0 | 0 | 1 | 1 | + | $0 \leq L \leq 2$ | $\|L-S\| \leq J \leq L+S$ |
| 2 | 2 | 0 | 0 | 0 | 2 | 0 | + | 2 | $\|2-S\| \leq J \leq 2+S$ |
| 2 | 0 | 2 | 0 | 0 | 0 | 2 | + | 2 | $\|2-S\| \leq J \leq 2+S$ |

Table of states for $N_{max} = 2$

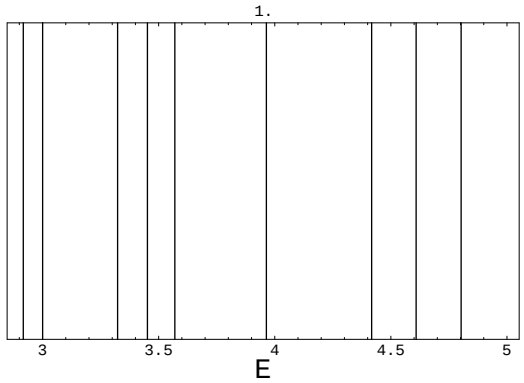
Energies for $\omega = 0.03$. The eigenvalues are distributed in four groups around the values $-3, -1, 1, 3$




Spectrum for $\omega = 0.1$ and $N = 2$,



## Spectrum for $\omega = 1$ and $N = 2$



# One-dimensional $n$ particles

The kinetic part of the hamiltonian

$$H = (\mathbf{1} + B) \sum_{i}^{n} \sigma_1^i a_i' + h.c. + \text{mass} \tag{11}$$

is infinitely degenerate. We have removed all other degrees of freedom in order to show that the cockroach nest makes itself present for an arbitrary number of interacting particles. Its elimination is not a trivial task, in despite of our careful choice of observables. To see the infinite degeneracy, apply eigenstates of $\sigma_1^i$ to $(H - \text{mass})^2$. Question: is there a system of one-dimensional Dirac particles which parallels the Calogero model?

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The original application of the three particle Dirac oscillator contained eigenstates of the permutation group, dealing also with particle statistics. At the end of the calculations, energies and eigenfunctions were obtained and a prediction for the form factor of the proton was given. However, let me quote Moshinsky once again

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# Exactly solvable extensions

Consider a hermitean operator of the form $\Phi(\mathbf{r},\mathbf{p})$ as the potential to be introduced in the total hamiltonian. One has $H^{(d)}=H_0^{(d)}+\Phi$, with $H_0^{(d)}$ given by the $d-$dimensional Dirac oscillator. On physical grounds, this corresponds to a bound fermion perturbed by a momentum-dependent potential. We introduce also an internal group for this field, for example the $SU(2)$ associated to isospin or as the gauge group of a non-abelian field

$$\Phi=\left(T_+\mathbf{S}\cdot\mathbf{a}+T_-\mathbf{S}\cdot\mathbf{a}^\dagger+\gamma T_3\right) \tag{12}$$

One may consider any potential of the form $\Phi=F(T_+\mathbf{S}\cdot\mathbf{a}+T_-\mathbf{S}\cdot\mathbf{a}^\dagger+\gamma T_3)$ where $F$ admits a power expansion. Evidently, $[I^{(3)},\Phi]=0$. A suitable group of states can be used to evaluate the $4\times 4$ blocks of $H$. We describe this procedure by restricting ourselves to the linear case for simplicity.

# Exactly solvable extensions

Lower dimensional examples follow the same pattern

$$H^{(1)} = \sigma_+ a + \sigma_- a^\dagger + m\sigma_3 + (A + \sigma_3 B)\left(T_+ a + T_- a^\dagger + \gamma T_3\right) \tag{13}$$

$$H^{(2)} = \sigma_+ A_R + \sigma_- A_R^\dagger + m\sigma_3 + (A + \sigma_3 B)\left(T_+ A_R + T_- A_R^\dagger + \gamma T_3\right) \tag{14}$$

$$H^{(3)} = \Sigma_+ \mathbf{S}\cdot\mathbf{a} + \Sigma_- \mathbf{S}\cdot\mathbf{a}^\dagger + m\Sigma_3 + (A + \Sigma_3 B)\left(T_+ \mathbf{S}\cdot\mathbf{a} + T_- \mathbf{S}\cdot\mathbf{a}^\dagger + \gamma T_3\right). \tag{15}$$

# Exactly solvable extensions

With these extensions, it is evident that the new invariants for one, two and three dimensions are

$$I^{(1)} = a^\dagger a + \frac{1}{2}\sigma_3 + \frac{1}{2}T_3 \tag{16}$$

$$I^{(2)} = A_R A_R^\dagger + \frac{1}{2}\sigma_3, \quad J_3 + \frac{1}{2}T_3 = A_R A_R^\dagger - A_L A_L^\dagger + \frac{1}{2}\sigma_3 + \frac{1}{2}T_3 \tag{17}$$

$$I^{(3)} = \mathbf{a}^\dagger \cdot \mathbf{a} + \frac{1}{2}\Sigma_3 + \frac{1}{2}T_3, \quad \mathbf{J} = \mathbf{a}^\dagger \times \mathbf{a} + \mathbf{S}. \tag{18}$$

# Exactly solvable extensions

Eigenstates of $H^{(3)}$. We proceed to evaluate the $4 \times 4$ matrix $H(N,j) \equiv \langle \quad | H^{(3)} | \quad \rangle$.

$$
\begin{aligned}
|\phi_1^N\rangle &= |n,(j+1/2,1/2)j,m_j\rangle|-\rangle_\Sigma|-\rangle_T \\
|\phi_2^N\rangle &= |n,(j-1/2,1/2)j,m_j\rangle|-\rangle_\Sigma|+\rangle_T \\
|\phi_3^N\rangle &= |n-1,(j-1/2,1/2)j,m_j\rangle|+\rangle_\Sigma|-\rangle_T \\
|\phi_4^N\rangle &= |n-1,(j+1/2,1/2)j,m_j\rangle|+\rangle_\Sigma|+\rangle_T
\end{aligned}
\tag{19}
$$

where $n$ is the oscillator radial number, $j$ is the total angular momentum and $m_j$ its projection in the $z$ axis. These are eigenstates of $I^{(3)}$ with eigenvalue $N = 2n + j - 1/2$.

# Exactly solvable extensions

The resulting $4 \times 4$ blocks of $H$ with elements $H(N,j)_{kl} = \langle \phi_k^N | H | \phi_l^N \rangle$ are

$$\begin{pmatrix} -m-(A-B)\gamma & (A-B)\sqrt{2(n+j)} & -\sqrt{2(n+j)} & 0 \\ (A-B)\sqrt{2(n+j)} & -m+(A-B)\gamma & 0 & \sqrt{2n} \\ -\sqrt{2(n+j)} & 0 & m-(A+B)\gamma & (A+B)\sqrt{2n} \\ 0 & \sqrt{2n} & (A+B)\sqrt{2n} & m+(A+B)\gamma \end{pmatrix}$$

and the secular equation $|H(N) - E| = 0$ can be solved explicitly using the formula for the roots of a quartic polynomial. The infinite degeneracy is now broken, since one cannot reduce $H(N)$ to smaller blocks where only $n$ appears. The exception to this occurs when $A = B = 0$, which obviously recovers the usual Dirac oscillator.

# Exactly solvable extensions

With the aid of the vector $u_\mu$ we can introduce more interactions in a covariant way. A non-local, non-abelian field tensor $\mathcal{F}^{\mu\nu} = \sum_{i=1}^{3} T_i \mathcal{F}_i^{\mu\nu}$ can be introduced in the equation by means of the Pauli coupling. We propose

$$\mathcal{F}_1^{\mu\nu} = \epsilon^{\mu\nu\lambda\rho} u_\lambda r_{\perp\rho} \tag{20}$$

$$\mathcal{F}_2^{\mu\nu} = \epsilon^{\mu\nu\lambda\rho} u_\lambda p_{\perp\rho} \tag{21}$$

$$\mathcal{F}_3^{\mu\nu} = 0, \tag{22}$$

for which the Dirac equation reads

$$[\gamma_\mu p^\mu + m + S_{\mu\nu} F^{\mu\nu} + B S_{\mu\nu} \mathcal{F}^{\mu\nu}]\psi = 0, \tag{23}$$

# Exactly solvable extensions

The nature of such field can be elucidated by inserting our $\mathcal{F}_{\mu\nu}$ in the corresponding non-local field equations. Using

$$\mathcal{F}^{\mu\nu} = u^{\mu}(r_{\perp}^{\nu} T_1 + p_{\perp}^{\nu} T_2) - \mu \leftrightarrow \nu \tag{24}$$

one has

$$\mathcal{F}^{\mu\nu} = i([p^{\mu}, B^{\nu}] - \mu \leftrightarrow \nu) + [B_{\mu}, B_{\nu}]. \tag{25}$$

# Exactly solvable extensions

The gauge potential and the current can be obatained in the form

$$B_\mu = u_\mu(\frac{1}{2} r_\mu r_\perp^\nu T_1 + r_\nu p_\perp^\nu T_2) \qquad \text{Bilinear in } p,\, r. \tag{26}$$

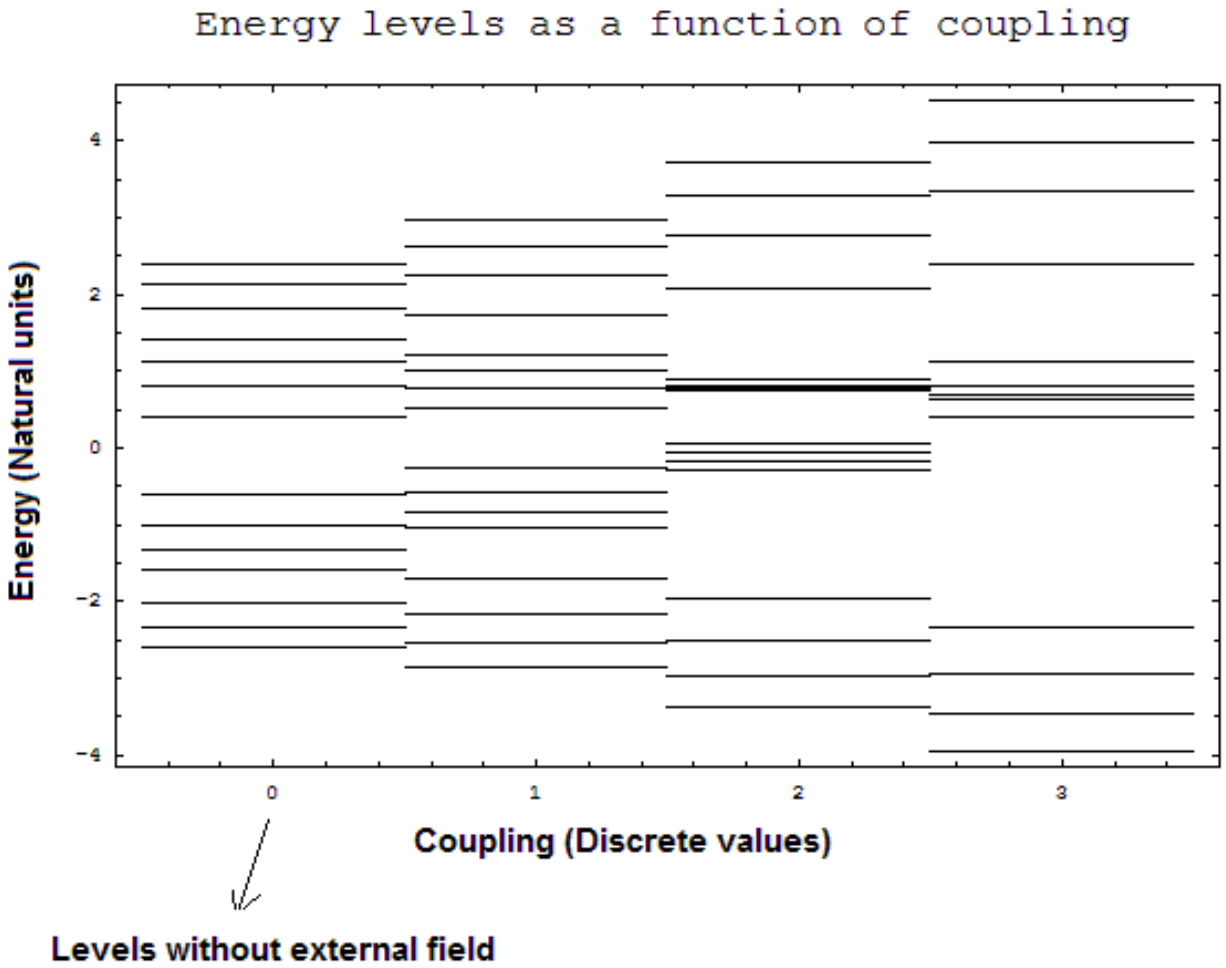
$$j^\nu = i[p_\mu, \tilde{\mathcal{F}}^{\mu\nu}] + [B_\mu, \tilde{\mathcal{F}}^{\mu\nu}] \tag{27}$$

$$= -u^\nu T_1 + p_\perp^\nu + \left(\frac{1}{2}\{p_\perp^\nu, r_\mu r_\perp^\mu\} - \{p_\perp^\mu, r_\perp^\nu\} r_\mu\right) T_2 \tag{28}$$

$$= -u^\nu T_1 + p_\perp^\nu + \text{ trilinear terms in } p,r\ .$$

# Eigenvalues

The vanishing coupling shows the eigenvalues of the Dirac oscillator. Degeneracies are lifted and level spacing increases.



# Quantum Optics

The structure of our hamiltonian shows that our model can be mapped to a Jaynes-Cummings hamiltonian of two atoms (of two levels each)

$$H = \sigma_+ a + \sigma_- a^\dagger + m\sigma_3 + T_+ a + T_- a^\dagger + \gamma T_3 \tag{29}$$

where $\sigma, T$ are now the operators for the atoms 1 and 2. The operator $a$ is the anhilation operator of the electromagnetic field mode. Spin-spin interactions can be introduced as well.

# Purity

Defining a partition of the system $A + B$
We take a pure state density operator $\rho = |\psi(t)\rangle\langle\psi(t)|$ of the entire system and compute purity $P$ and entropy $S$ of the Dirac oscillator subsystem.

$$\begin{aligned} P(t) &= \mathrm{Tr}_{\mathrm{N},\sigma}\left(\left(\mathrm{Tr}_{\tau}\rho(\mathrm{t})\right)^2\right) \\ S(t) &= -\mathrm{Tr}_{\mathrm{N},\sigma}\left(\mathrm{Tr}_{\tau}\rho(\mathrm{t})\mathrm{Log}\left(\mathrm{Tr}_{\tau}\rho(\mathrm{t})\right)\right), \end{aligned} \tag{30}$$

where $\mathrm{Tr}_{\mathrm{N},\sigma}$ is the trace with respect to oscillator and $*$-spin degrees of freedom, while $\mathrm{Tr}_{\tau}$ is the trace with respect to isospin.

# Integrability

Integrals of the motion

$$I^{(3)} = \mathbf{a}^\dagger \cdot \mathbf{a} + \frac{1}{2}\Sigma_3 + \frac{1}{2}T_3, \quad \mathbf{J} = \mathbf{a}^\dagger \times \mathbf{a} + \mathbf{S}. \tag{31}$$

but we analyze the one dimensional case for simplicity

$$I^{(1)} = a^\dagger a + \frac{1}{2}\sigma_3 + \frac{1}{2}T_3 \tag{32}$$

# Integrability

We use the eigenstates of $I^{(1)}$

$$|\phi_1^n\rangle = |n+2\rangle|--\rangle \qquad |\phi_2^n\rangle = |n+1\rangle|-+\rangle$$
$$|\phi_3^n\rangle = |n+1\rangle|+-\rangle \qquad |\phi_4^n\rangle = |n\rangle|++\rangle \tag{33}$$

$$H = \begin{pmatrix} H_0 & 0 & 0 & \dots \\ 0 & H_1 & 0 & \dots \\ 0 & 0 & H_2 & \\ \vdots & \vdots & & \ddots \end{pmatrix}, \tag{34}$$

where $H_n$ is a $4 \times 4$ block.

# Entanglement with external fields

Initial state $\psi = \chi_n \otimes \chi$,

$$|\chi\rangle = 1/\sqrt{2}(\cos\theta|+\rangle + \sin\theta|-\rangle) \tag{35}$$

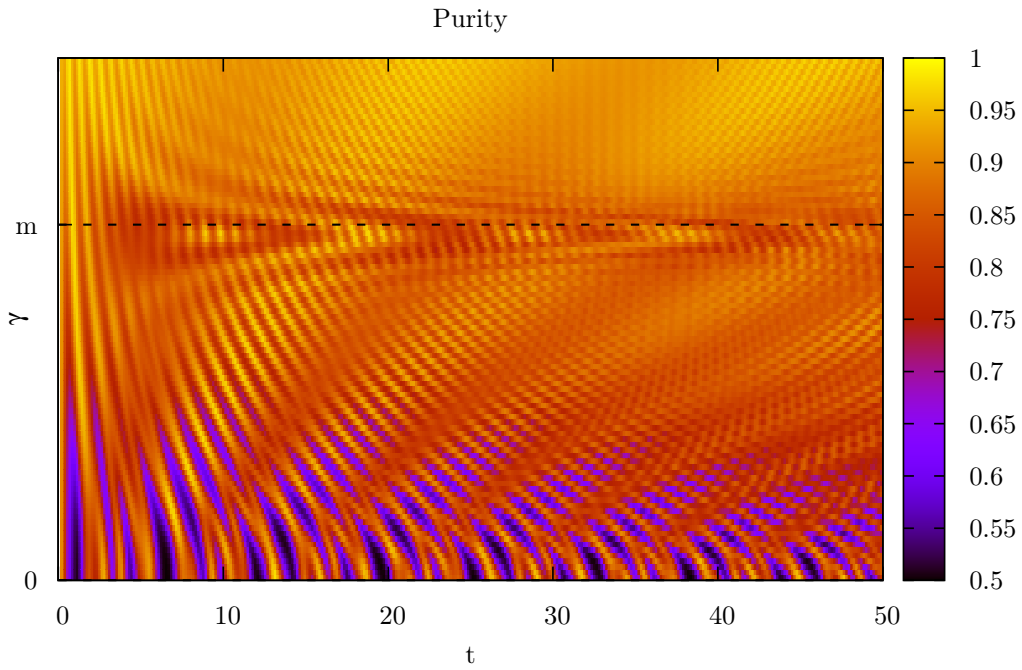
and $\chi_n$ is a solution of the unperturbed Dirac oscillator

$$|\chi_n\rangle = A_n^{(+)}|n\rangle|+\rangle + A_n^{(-)}|n+1\rangle|-\rangle \tag{36}$$

We use exact solutions to compute $P(t), S(t)$ (purity and entropy). Other initial conditions can be used in the context of Quantum optics, but this side of the analogy is not discussed here.

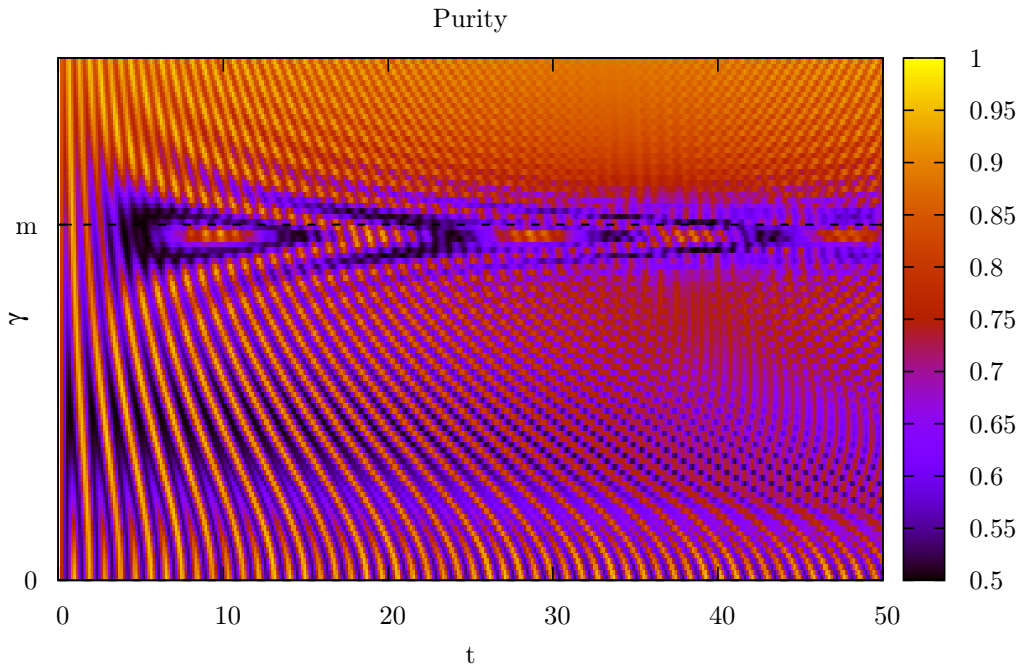
# Results

A resonant effect around $\gamma = m$



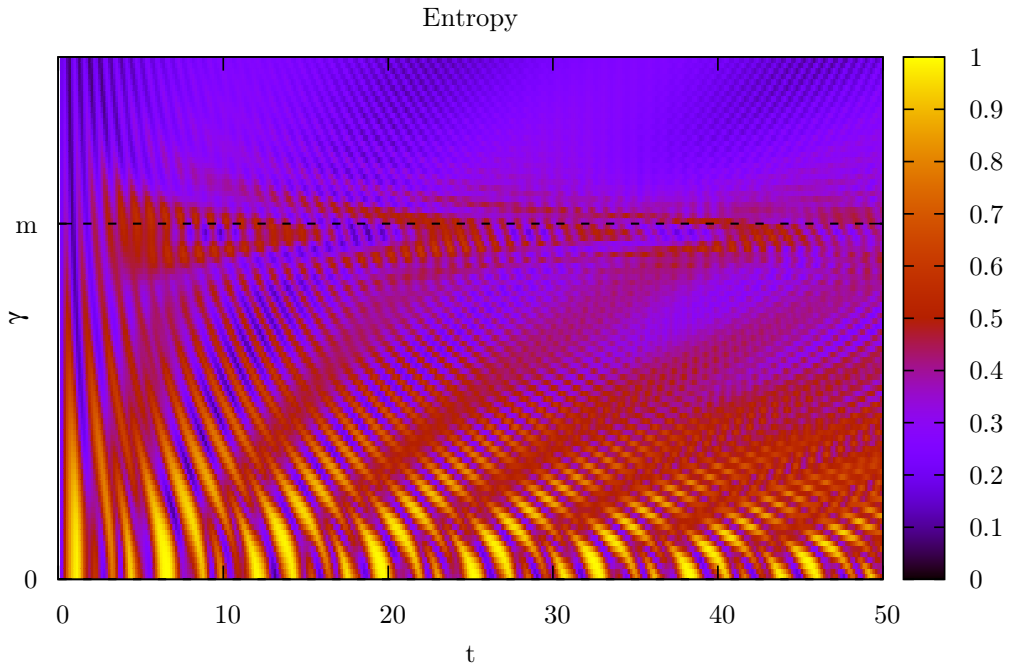
# Results

A resonant effect around $\gamma = m$



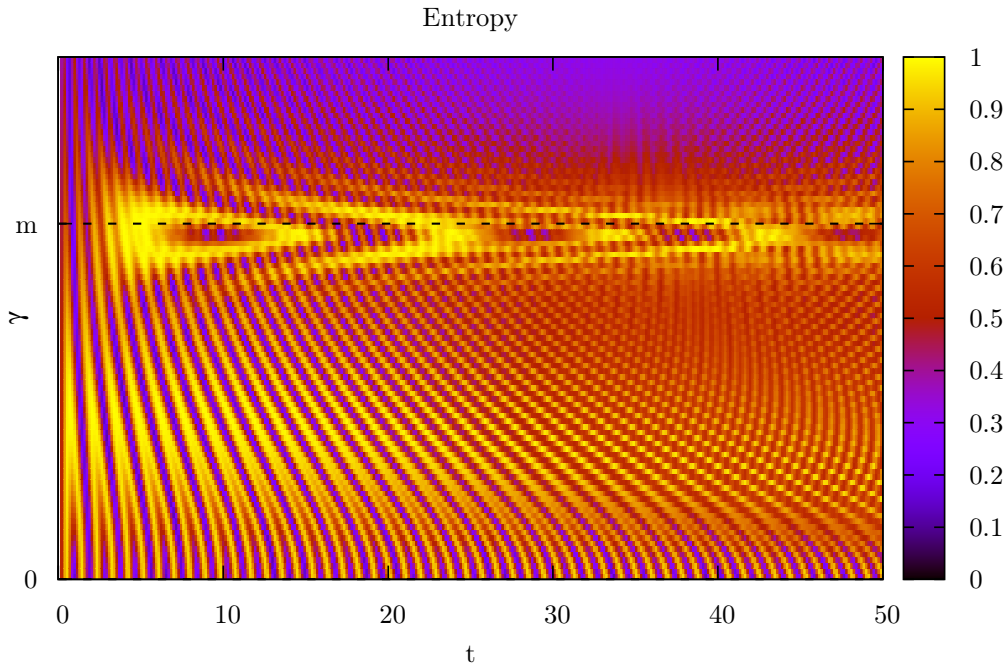
# Results

A resonant effect around $\gamma = m$



# Results

A resonant effect around $\gamma = m$



# Remarks

- Purity as a measure of relativistic entanglement requires a good choice of partitions
- A toy model suggests that particle creation and maximal entanglement are related
- This can be interpreted as a resonant effect in the Quantum Optics analogy

# The Dirac (Moshinsky) Oscillator: Theory and applications

ELAF July 29, 2010

E. Sadurní, Institut fuer Quantenphysik, Uni-Ulm

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# Contents

Emulating Graphene in Electromagnetic Billiards

- What is Graphene ?
- One-dimensional toy models
  - The free Dirac equation
  - A deformed lattice
- Two-dimensional lattices
  - The free case
  - The two-dimensional Dirac oscillator
- The importance of tight binding

# The Dirac oscillator

The equation leading to the Dirac oscillator hamiltonian is

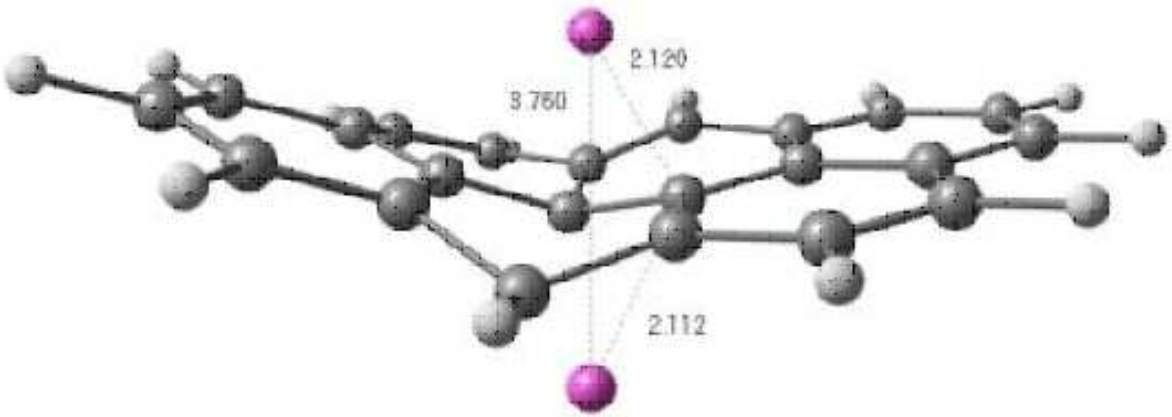
$$[\gamma_\mu(p^\mu + i\gamma_\nu u^\nu r_\perp^\mu) + m]\psi = 0, \tag{1}$$

but we shall use the one and two dimensional hamiltonians

$$H = \sigma_+ a + \sigma_- a^\dagger + m\sigma_3, \tag{2}$$

$$H = \sigma_+ a_R + \sigma_- a_R^\dagger + m\sigma_3 \tag{3}$$

# Our motivation: Graphene



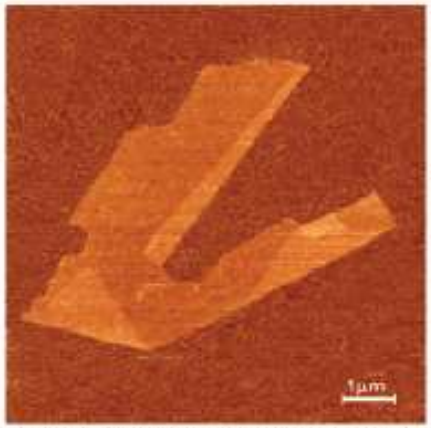




*Fig. 2 Atomic force microscopy image of a graphene crystal on top of an oxidized Si substrate. Folding of the flake can be seen. The measured thickness of graphene corresponds to the interlayer distance in graphite. Scale bar = 1 μm. (Reprinted with permission from[13]. © 2005 National Academy of Sciences.)*

# What is graphene?

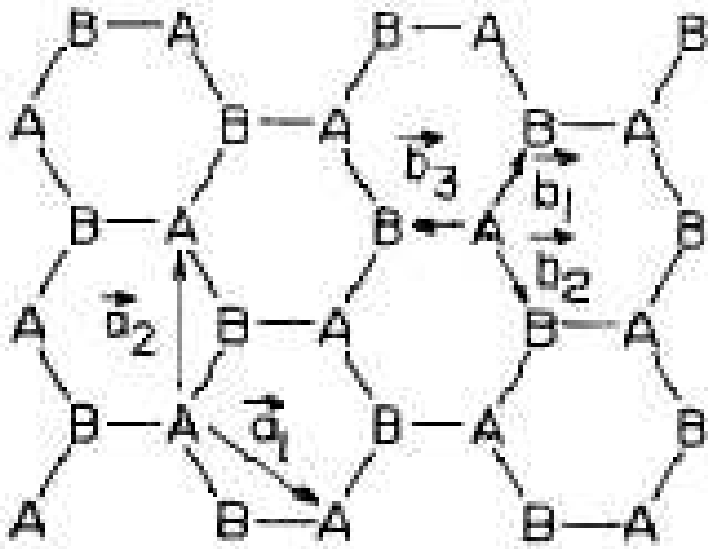
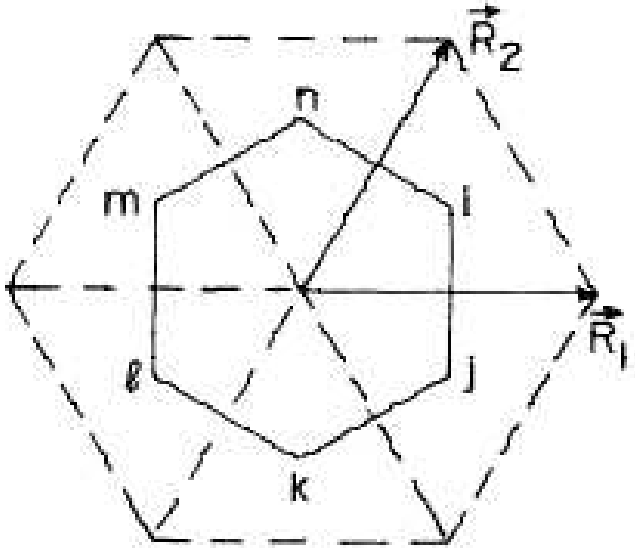



FIG. 1. The honeycomb lattice as a superposition of two triangular sublattices. The basis vectors are $\vec{a}_1 = (\sqrt{3}/2, -\frac{1}{2})a$; $\vec{a}_2 = (0, 1)a$ and the sublattices are connected by $\vec{b}_1 = (1/2\sqrt{3}, \frac{1}{2})a$; $\vec{b}_2 = (1/2\sqrt{3}, -\frac{1}{2})a$; $\vec{b}_3 = (-1/\sqrt{3}, 0)a$.

2450

# What is graphene?




identified sides

$ij \equiv \ell m$
$jk \equiv mn$
$k\ell \equiv ni$

equivalent corners

$i \equiv k \equiv m$
$j \equiv \ell \equiv n$

FIG. 2. The Brillouin zone. The reciprocal-lattice basis vectors are $\vec{R}_1 = (4\pi/\sqrt{3}a)(1,0)$; $\vec{R}_2 = (4\pi/\sqrt{3}a) \times (\frac{1}{2}, \frac{1}{2}\sqrt{3})$. The degeneracy points occur at the corners, *ijklmn*, of the Brillouin zone. Two of these are inequivalent; we have chosen $\vec{q}_1 = (4\pi/\sqrt{3}a)(\frac{1}{2}, 1/2\sqrt{3})$ at point *i* and $\vec{q}_2 = -\vec{q}_1$ at point *l*.

# History (1947-2007)

- Band theory in tight binding approximation (1946)

PHYSICAL REVIEW VOLUME 71, NUMBER 9 MAY 1, 1947

**The Band Theory of Graphite**

P. R. WALLACE*
*National Research Council of Canada, Chalk River Laboratory, Chalk River, Ontario*
(Received December 19, 1946)

The structure of the electronic energy bands and Brillouin zones for graphite is developed using the "tight binding" approximation. Graphite is found to be a semi-conductor with zero activation energy, i.e., there are no free electrons at zero temperature, but they are created at higher temperatures by excitation to a band contiguous to the highest one which is normally filled. The electrical conductivity is treated with assumptions about the mean free path. It is found to be about 100 times as great parallel to as across crystal planes. A large and anisotropic diamagnetic susceptibility is predicted for the conduction electrons; this is greatest for fields across the layers. The volume optical absorption is accounted for.

- Field theory of electrons in 2+1 from lattice (1984)

PHYSICAL REVIEW
LETTERS

VOLUME 53 24 DECEMBER 1984 NUMBER 26

**Condensed-Matter Simulation of a Three-Dimensional Anomaly**

Gordon W. Semenoff
*The Institute for Advanced Study, Princeton, New Jersey 08540, and Department of Physics,[a] University of British Columbia, Vancouver, British Columbia V6T 2A6, Canada*
(Received 4 September 1984)

A condensed-matter analog of (2+1)-dimensional electrodynamics is constructed, and the consequences of a recently discovered anomaly in such systems are discussed.

# History (1947-2007)

- Experimental evidence (2005)

Vol 438|10 November 2005|doi:10.1038/nature04233 nature

LETTERS

## Two-dimensional gas of massless Dirac fermions in graphene

K. S. Novoselov[1], A. K. Geim[1], S. V. Morozov[2], D. Jiang[1], M. I. Katsnelson[3], I. V. Grigorieva[1], S. V. Dubonos[2] & A. A. Firsov[2]

- Review of results: theory and experiment (Novoselov, 2007)
  - Spectrum measurement
  - Quantum Hall effect
  - Klein's Paradox
  - Topology and curvature deffects

# History (1947-2007)

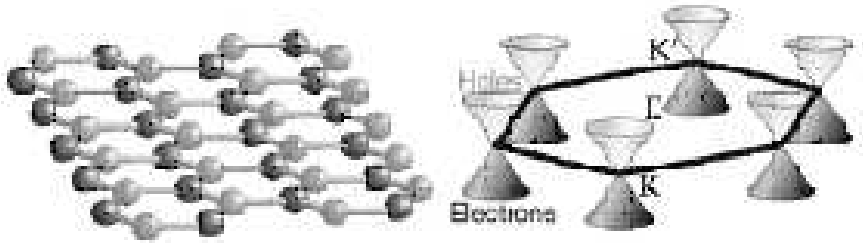
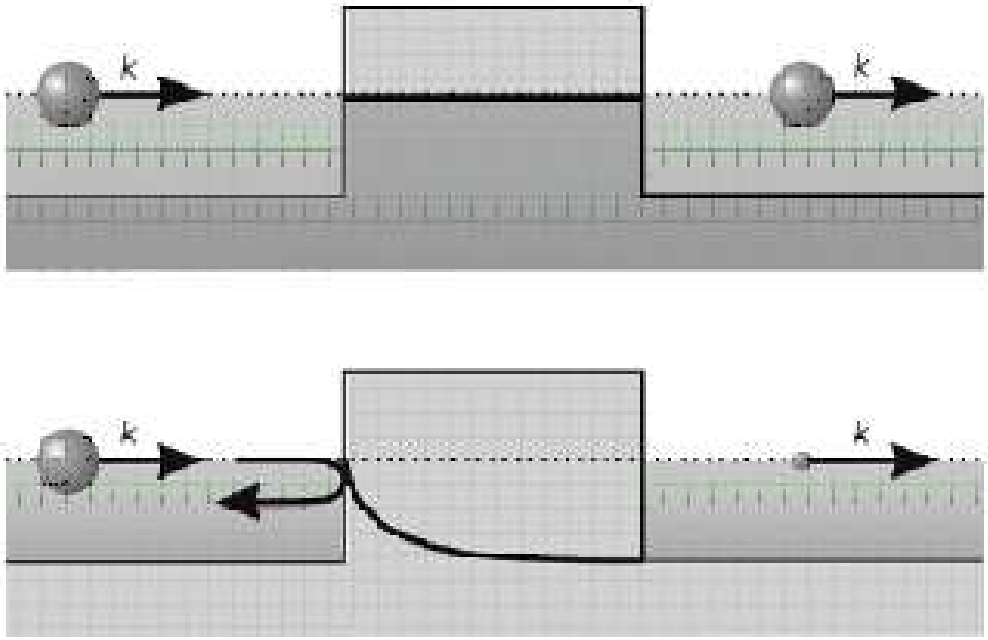



Fig. 1. Left: Crystallographic structure of graphene. Atoms from different sublattices (A and B) are marked by different shades of gray. Right: Band structure of graphene in the vicinity of the Fermi level. The conductance band touches the valence band at $K$ and $K'$ points.



# General considerations

- For our purposes, the situation can be modelled by a Schroedinger equation with a potential consisting of deep wells, each of them located at a lattice point. The specific shape of atomic wave functions is irrelevant, as long as we know how the overlaps (interactions) decay as a function of the distance between resonators. For practical purposes, such decay can be regarded as exponential, which follows from considering a lattice of constant potential wells. As an additional remark, such potentials should be deep enough such that only one level (or isolated resonance) well below the surface contributes to the dynamics.

# One dimensional model

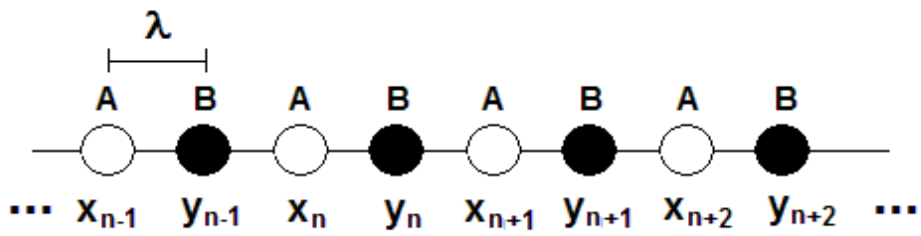
A lattice consisting of two periodic sublattices is considered. They have the same period and are denoted as type A and type B. Each sublattice point can be labeled by an integer $n$ according to its position on the line, *i.e.* $x_n$. The energy of the single level to be considered in the well is denoted by $\alpha$ for type A and $\beta$ for type B. The state corresponding to a particle in site $n$ of lattice A is denoted by $|n\rangle_A$ and the corresponding localized wave function is given by $\xi_A(x - x_n) = \langle x|n\rangle_A$. The same applies for B. The probability amplitude $\Delta$ (or overlap) between nearest neighbors is taken as a real constant.

$$H = \begin{pmatrix} H_{AA} & H_{AB} \\ H_{BA} & H_{BB} \end{pmatrix} \tag{4}$$

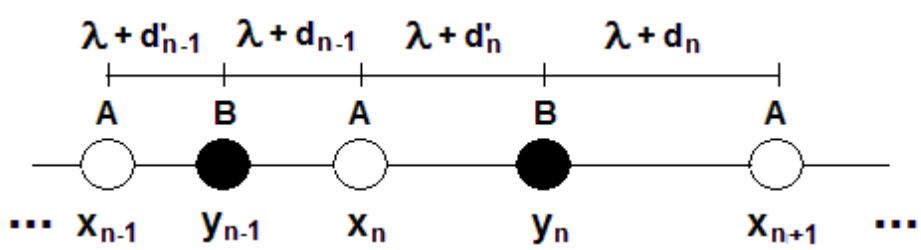
# One dimensional model

Configuration of potential wells (or resonators) on a chain. a) The periodic case. b) General deformation. c) Dimer deformation.

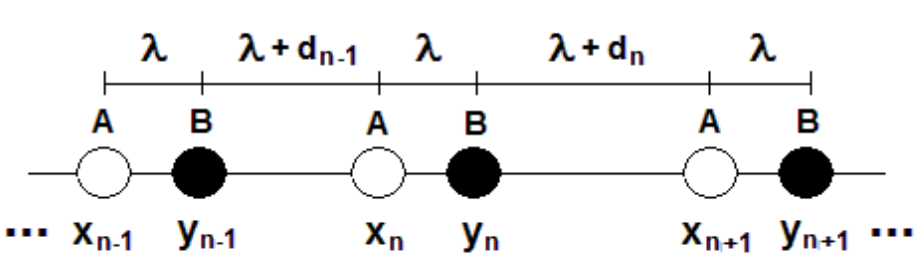
a)




b)

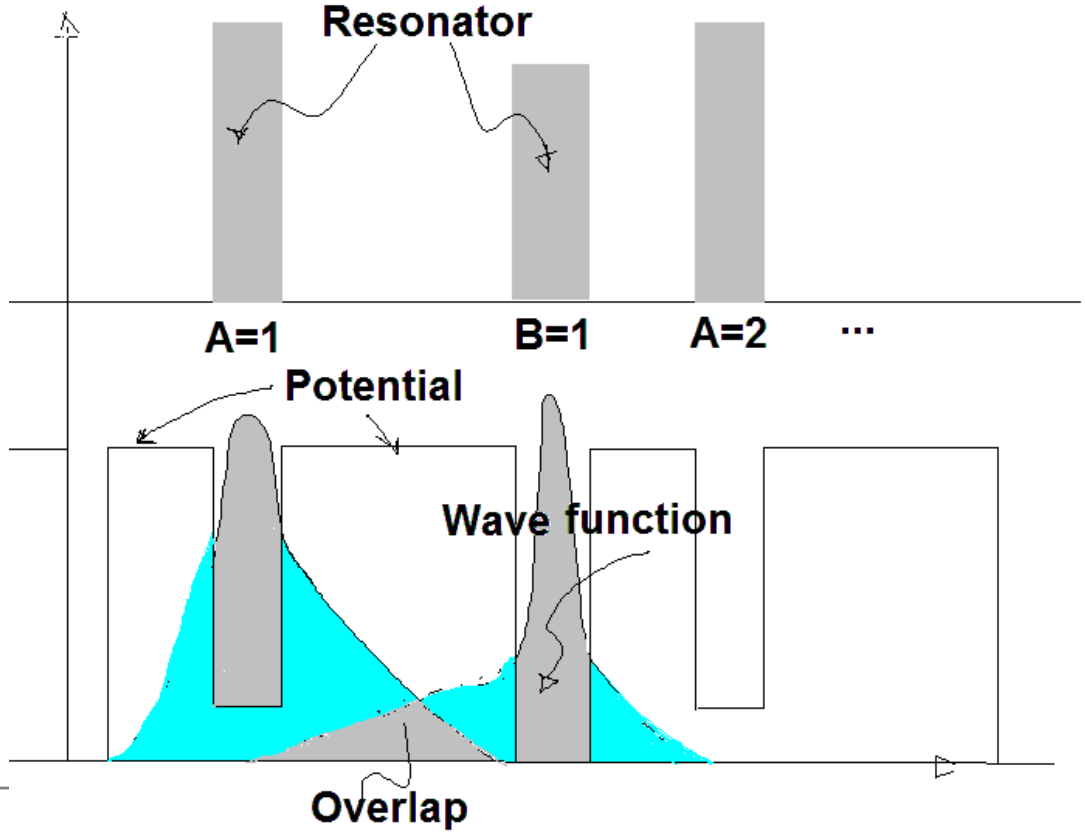



c)



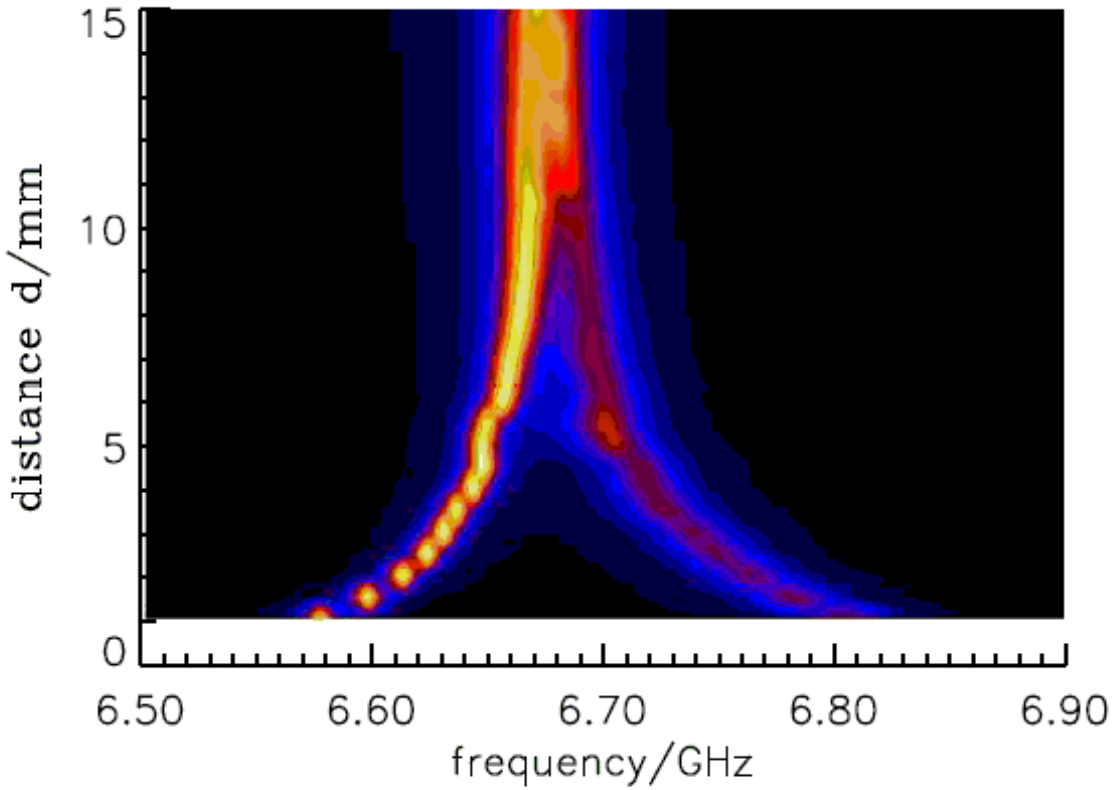
# One dimensional model

Resonators in a one dimensional lattice. The plot above gives a representation of resonators as a function of the x-coordinate, while the plot below shows an idealization of the corresponding potential (wells) and the wave functions of resonances. These functions may leak outside the wells.



# One dimensional model

Density plot of coupling between resonators. Exponential decay.



# The free case

The hamiltonian can be cast in terms of Pauli matrices $\sigma_3, \sigma_+ = \sigma_1 + i\sigma_2, \sigma_- = \sigma_+^\dagger$ by defining

$$\Pi = \begin{pmatrix} \ddots & & & \\ & \Delta & \Delta & \\ & & \Delta & \Delta \\ & & & \ddots \end{pmatrix} \tag{5}$$

and setting $M = (\alpha - \beta)/2, E_0 = (\alpha + \beta)/2$. We have

$$H = E_0 + \sigma_3 M + \sigma_+ \Pi + \sigma_- \Pi^\dagger \tag{6}$$

This is a general structure which explains the appearance of pseudospin.


It is left to show that there is a region where the spectrum is linear (Dirac). The spectrum is computed by squaring $H$.

$$(H - E_0)^2 = M^2 + \Pi\Pi^\dagger \tag{7}$$

Bloch's theorem enters in the form

$$\Pi\phi_k = \Delta(1 + e^{i2\pi\lambda k})\phi_k, \quad \Pi\Pi^\dagger\phi_k = \Delta^2|1 + e^{i2\pi\lambda k}|^2\phi_k \tag{8}$$

The energies and eigenfunctions of $H$ are

$$E(k) = E_0 \pm \sqrt{\Delta^2 |1 + e^{i2\pi\lambda k}|^2 + M^2} \tag{9}$$

$$\psi^{\pm} = N \begin{pmatrix} \phi_k \\ \frac{\pm E(k) - E_0 - M}{\Delta(1+e^{i2\pi\lambda k})} \phi_k \end{pmatrix}, \tag{10}$$

Around points where the inter-band distance is minimal, we have the usual relativistic formula

$$E(\kappa) = E_0 \pm \sqrt{\Delta^2 \kappa^2 + M^2}, \tag{11}$$


The amplitudes are proportional to the overlap between neighboring sites and decay exponentially as a function of the separation distance between resonators, *i.e.*

$$\Delta_{n,n+1} = \Delta e^{-d_n/\Lambda}, \tag{12}$$

where $d_n + \lambda$ is the separation distance between resonators of type A and B in the $n$-th position. When $d_n = 0$, the periodic configuration is recovered. The length $\Lambda$ has been introduced for phenomenological reasons: The decay law might be given by a multipole law, but we fit it to an exponential decay by adjusting $\Lambda$.

With all this, it is natural to expect a modification in the operators $\Pi, \Pi^\dagger$. We use $a, a^\dagger$ and impose $[a, a^\dagger] = \omega = constant$ (The limit $\omega = 0$ recovers Bloch's theorem). One finds the conditions

$$\Delta_{n,n} = \Delta, \qquad \Delta^2_{n+1,n+2} - \Delta^2_{n,n+1} = \omega \tag{13}$$

Therefore the distance formula for the resonators is

$$d_n = \Lambda \log\left(\frac{\Delta^2}{\Delta^2 - n\omega}\right), \qquad 0 < n < n_{max} \tag{14}$$

with $n_{max} = [|\frac{\Delta^2}{\omega}|]$.

# One dimensional Dirac oscillator

Finally, we have the hamiltonian

$$H = E_0 + \sigma_3 M + \sigma_+ a + \sigma_- a^\dagger \tag{15}$$

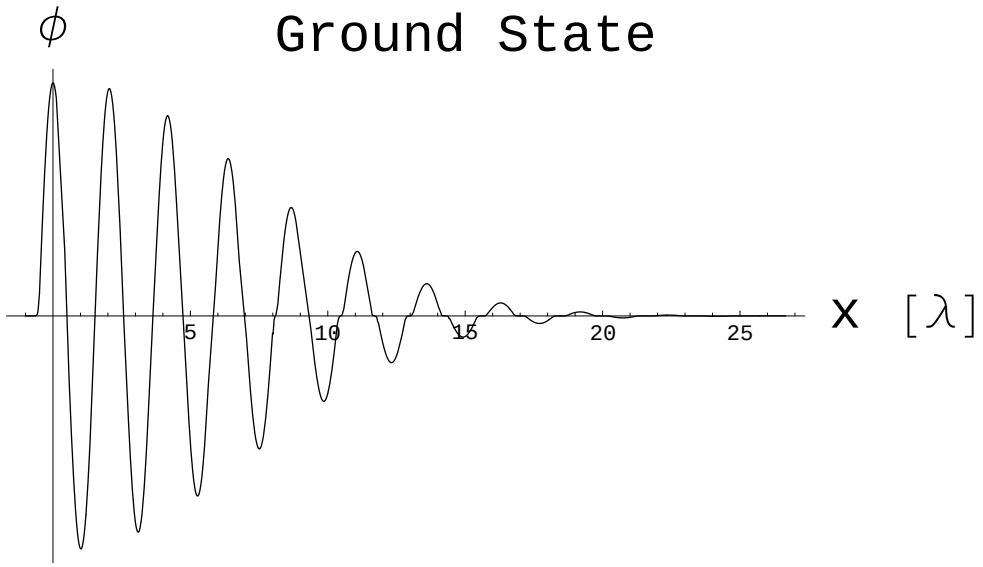
with energies and wave functions

$$E(n) = E_0 \pm \sqrt{\omega n + M^2}, \qquad 0 > n > \Delta^2/\omega \tag{16}$$

$$\psi^\pm = N \begin{pmatrix} \phi_{n+1} \\ \frac{\pm(E(n)-E_0)-M}{\sqrt{\omega(n+1)}}\phi_n \end{pmatrix},$$

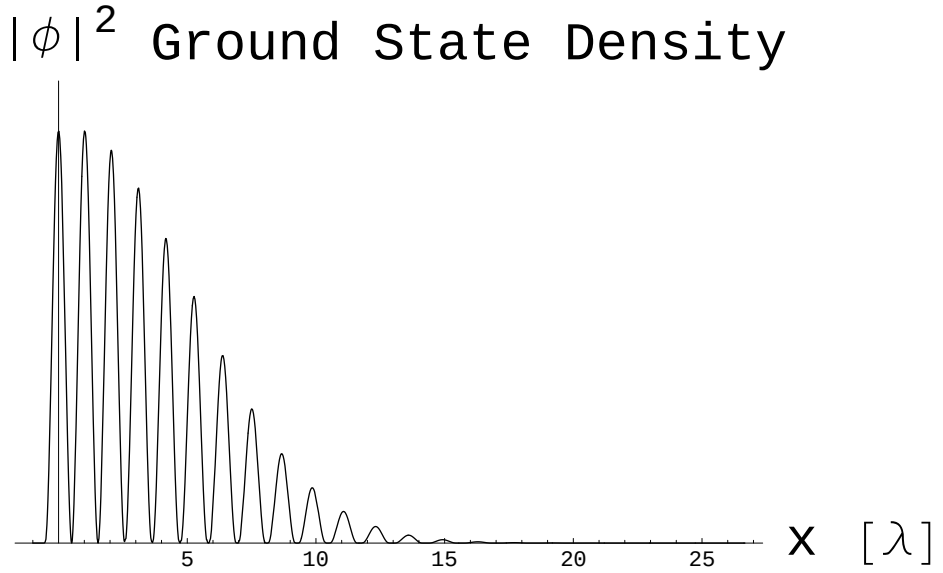
# Wavefunctions

Ground state as a function of site number. The ground state wavefunction is obtained by multiplying the values given in the ordinate by the individual resonant wavefunctions. These are considered to be highly peaked at each site. The signs alternate from site to site.



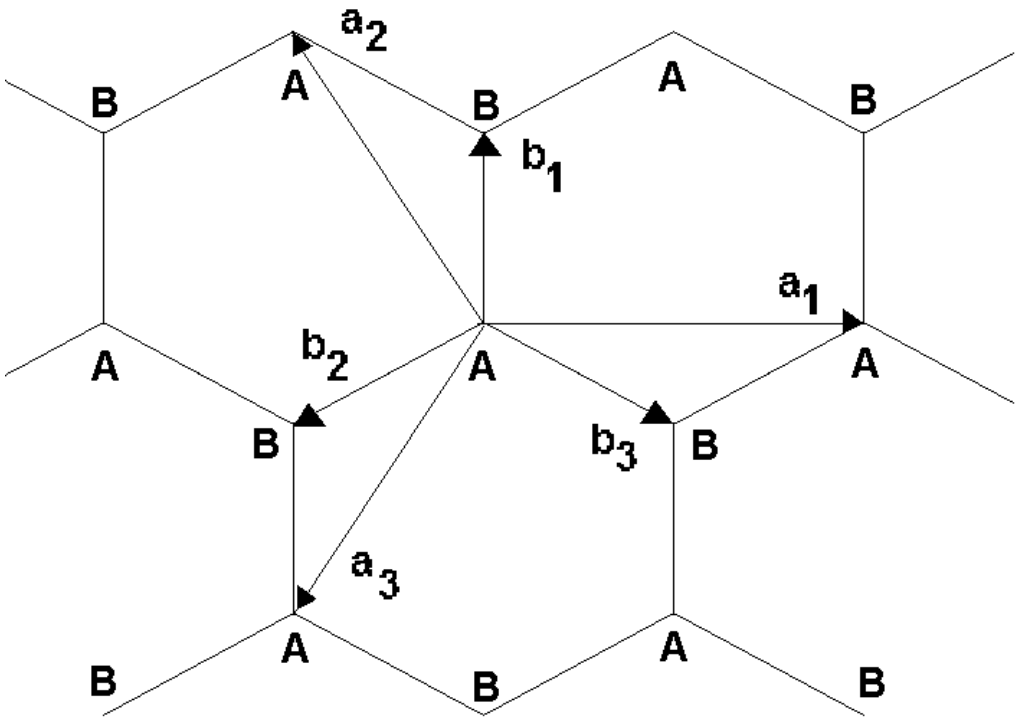
# Wavefunctions

Ground state density as a function of site number. The probability density is obtained by multiplying the values in the ordinate by the individual resonant wavefunctions, which are considered to be highly peaked at each site. The density does not exhibit nodes.



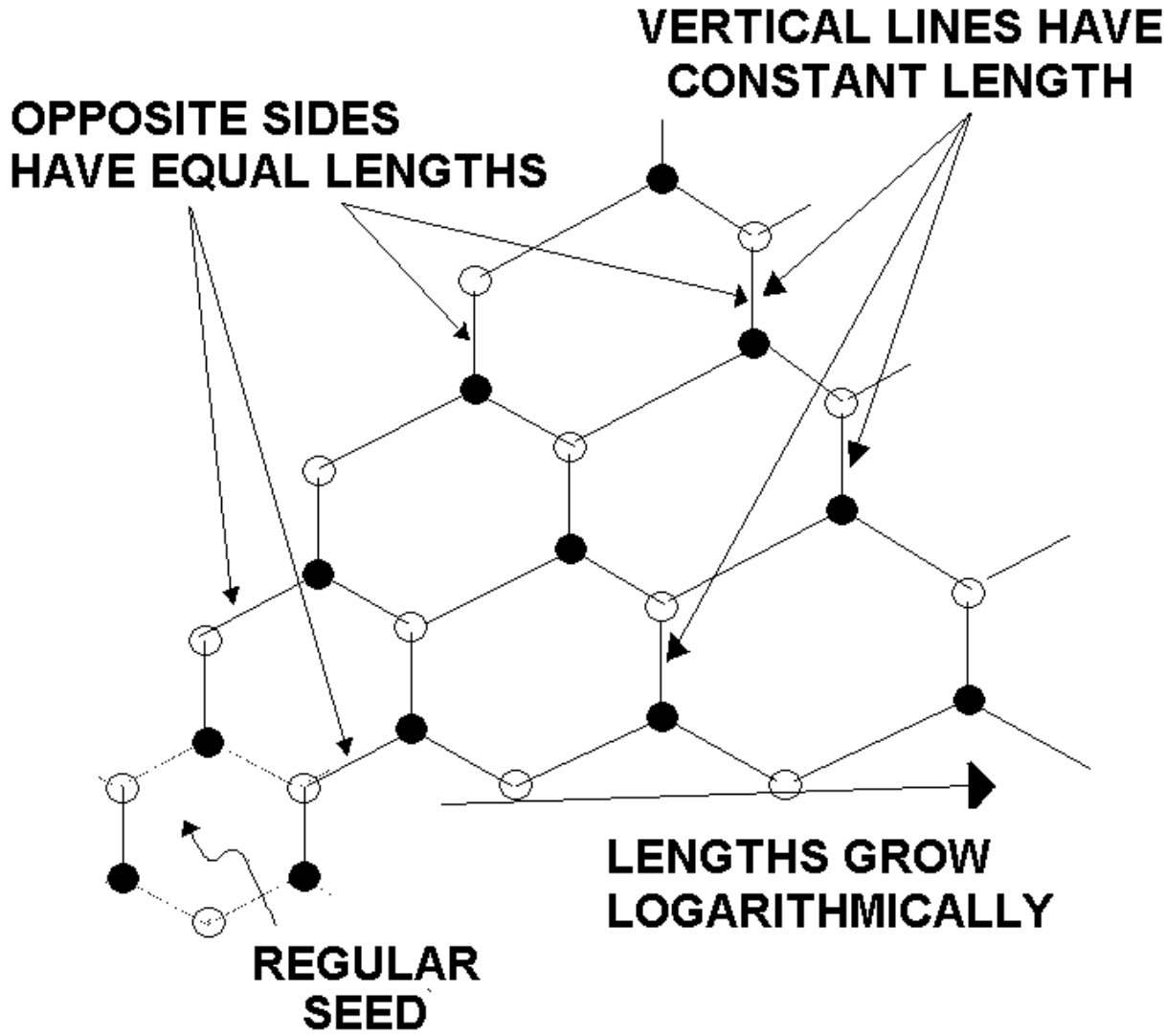
# Two dimensional lattice

The concepts given in the last sections are now extended to produce an emulation of graphene. We shall use the same algabraic strategy to derive spectra and a possible extension through deformations, namely the two dimensional Dirac oscillator.

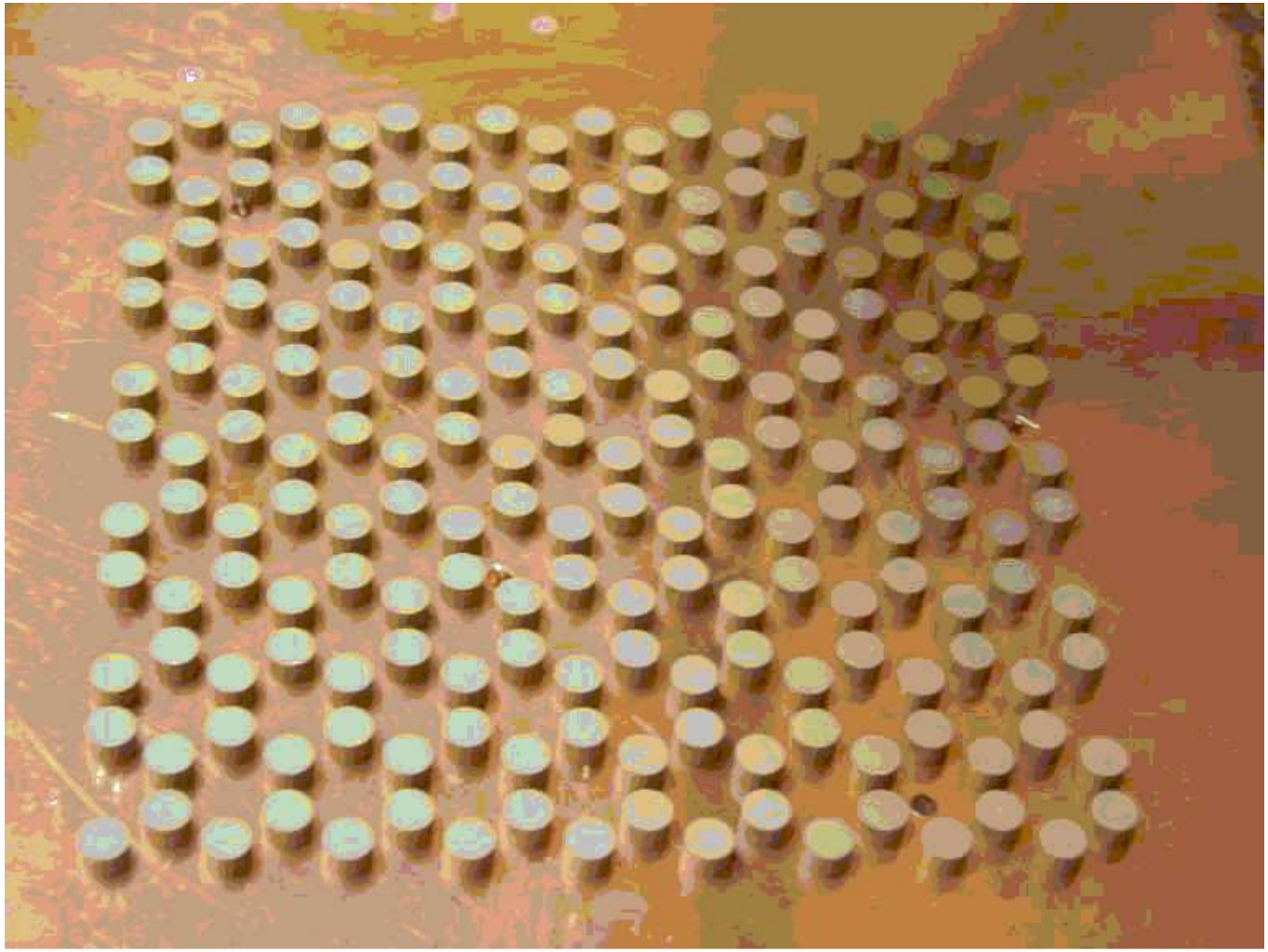


# Two dimensional lattice

Deformations



# Two dimensional lattice



# The free case in 2D

We start with the tight binding hamiltonian

$$\begin{aligned} H &= \alpha \sum_{\mathbf{A}} |\mathbf{A}\rangle\langle\mathbf{A}| + \beta \sum_{\mathbf{A}} |\mathbf{A}+\mathbf{b}_1\rangle\langle\mathbf{A}+\mathbf{b}_1| \\ &+ \sum_{\mathbf{A},i=1,2,3} \Delta \left( |\mathbf{A}\rangle\langle\mathbf{A}+\mathbf{b}_i| + |\mathbf{A}+\mathbf{b}_i\rangle\langle\mathbf{A}| \right) \end{aligned} \tag{17}$$

The usual Pauli operators are constructed through the definitions

$$\sigma_+ = \sum_{\mathbf{A}} |\mathbf{A}\rangle\langle\mathbf{A}+\mathbf{b}_1|, \qquad \sigma_- = \sigma_+^\dagger \tag{18}$$

$$\sigma_3 = \sum_{\mathbf{A}} |\mathbf{A}\rangle\langle\mathbf{A}| - |\mathbf{A}+\mathbf{b}_1\rangle\langle\mathbf{A}+\mathbf{b}_1|, \tag{19}$$


while the kinetic operators $\Pi, \Pi^\dagger$ are defined as

$$\Pi = \sum_{\mathbf{A},i} \Delta \left( |\mathbf{A}\rangle\langle \mathbf{A} + \mathbf{b}_i - \mathbf{b}_1| + |\mathbf{A} + \mathbf{b}_1\rangle\langle \mathbf{A} + \mathbf{b}_i| \right). \tag{20}$$

The spectrum and eigenfunctions are obtained again by squaring $H$. With $M$ and $E_0$ given as before, we obtain

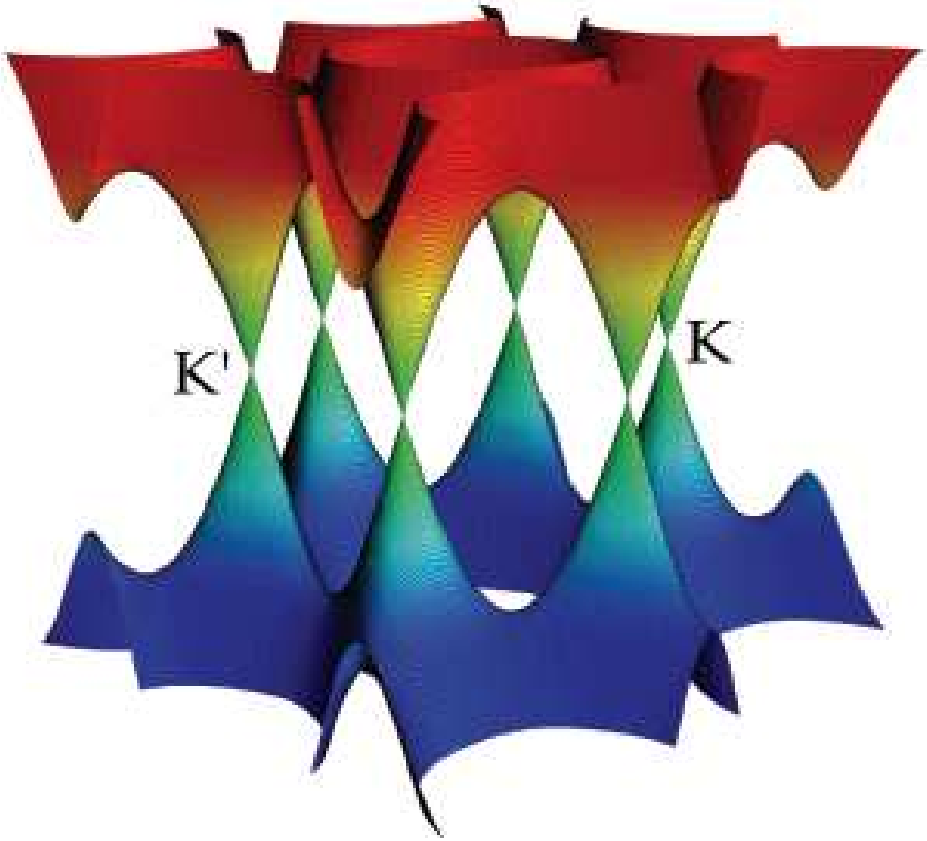
$$H = E_0 + M\sigma_3 + \sigma_+\Pi + \sigma_-\Pi^\dagger \tag{21}$$

and

$$(H - E_0)^2 = M^2 + \Pi\Pi^\dagger \tag{22}$$

# The Dirac points

Energy surfaces (taken from Novoselov et al.)



# The Dirac points

The spectrum and eigenfunctions are then

$$E(\mathbf{k}) = E_0 \pm \sqrt{\Delta^2 |\sum_i e^{i2\pi\lambda \mathbf{b}_i \cdot \mathbf{k}}|^2 + M^2} \tag{23}$$

$$\psi^{\pm} = C^{\pm}\phi_k^1 + D^{\pm}\phi_k^2, \qquad C^{\pm} = \frac{\pm(E(\mathbf{k}) - E_0) - M}{\Delta(\sum_i e^{i2\pi\lambda \mathbf{b}_i \cdot \mathbf{k}})} D^{\pm} \tag{24}$$

It is well known that the degeneracy points of the spectrum for the massless case are $\mathbf{k}_0 = \pm\frac{1}{2\lambda}(1, -\sqrt{3})$. Around such points one finds

$$E(\mathbf{k} - \mathbf{k}_0) - E_0 = \pm\sqrt{\Delta^2 k^2 + M^2} \tag{25}$$

# The Dirac oscillator in 2D

We deform the lattice through an extension of the kinetic operators, just as in the one dimensional case. Let us consider site dependent transition amplitudes $\Delta(\mathbf{A}, \mathbf{A}+\mathbf{b}_1)$ connecting the sites labeled by $\mathbf{A}, \mathbf{A}+\mathbf{b}_1$. Again, these are related to distances $d(\mathbf{A}, \mathbf{A}+\mathbf{b}_1)$ between resonators as $\Delta(\mathbf{A}, \mathbf{A}+\mathbf{b}_1) = \Delta \exp(-d(\mathbf{A}, \mathbf{A}+\mathbf{b}_1)/\Lambda)$. Now we define the ladder operator

$$a_R = \sum_{\mathbf{A},i} \Delta(\mathbf{A}, \mathbf{A}+\mathbf{b}_1) \left( |\mathbf{A}\rangle\langle \mathbf{A}+\mathbf{b}_i - \mathbf{b}_1| + |\mathbf{A}+\mathbf{b}_1\rangle\langle \mathbf{A}+\mathbf{b}_i| \right) \qquad (26)$$

and impose $[a_R, a_R^\dagger] = \omega$.


After some algebra, one can prove that this leads to recurrence relations

$$\Delta(\mathbf{A}, \mathbf{A}+\mathbf{b}_1) = \Delta, \tag{27}$$

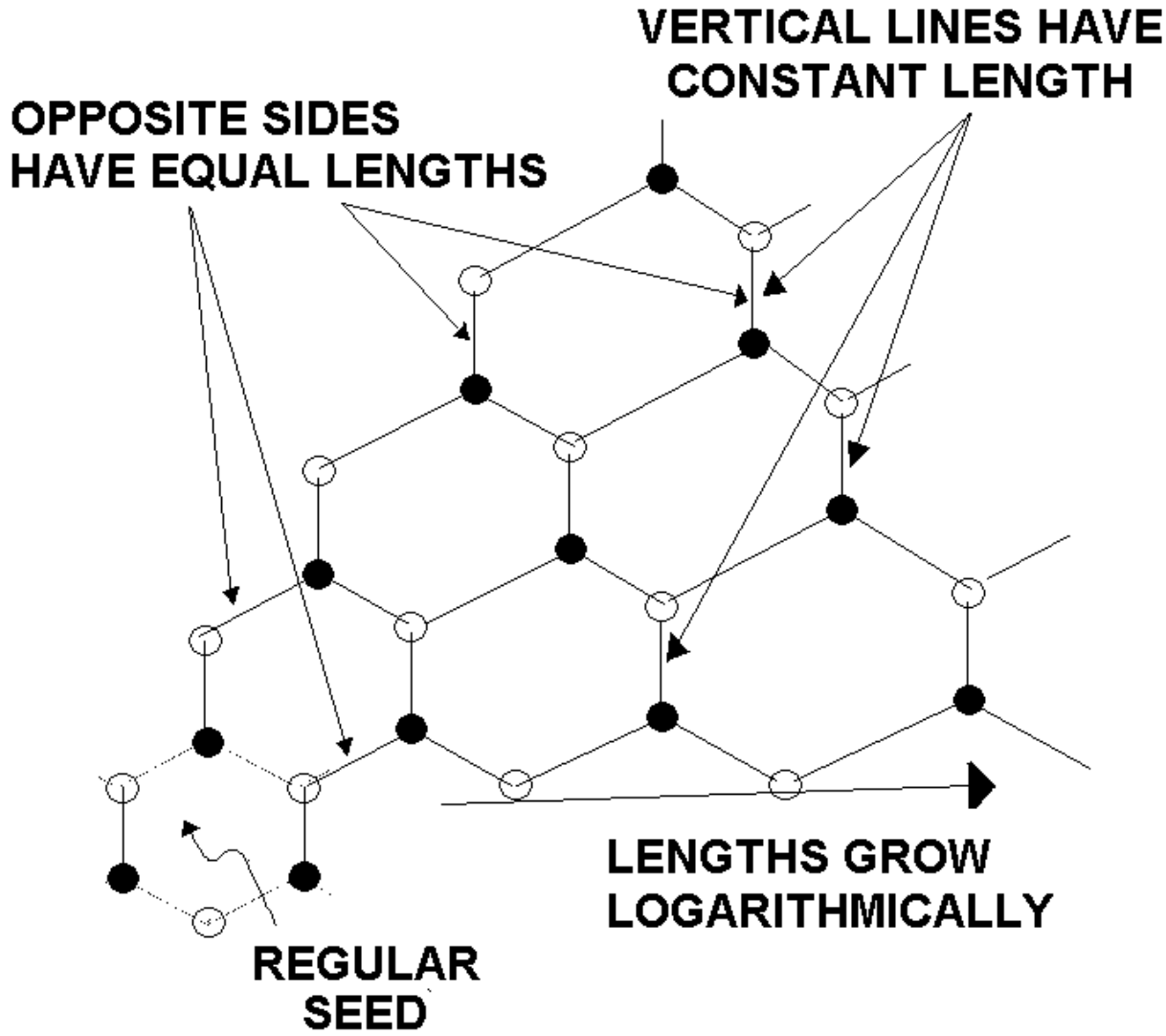
$$\Delta^2(\mathbf{A}, \mathbf{A}+\mathbf{b}_2) + \Delta^2(\mathbf{A}+\mathbf{b}_2, \mathbf{A}+\mathbf{b}_2-\mathbf{b}_3) = \tag{28}$$
$$\Delta^2(\mathbf{A}+\mathbf{b}_1, \mathbf{A}+\mathbf{b}_1+\mathbf{b}_2) + \Delta^2(\mathbf{A}+\mathbf{b}_2+\mathbf{b}_1, \mathbf{A}+\mathbf{b}_1+\mathbf{b}_2-\mathbf{b}_3),$$

$$\Delta^2(\mathbf{A}, \mathbf{A}+\mathbf{b}_2) + \Delta^2(\mathbf{A}, \mathbf{A}+\mathbf{b}_3) = \tag{29}$$
$$\Delta^2(\mathbf{A}+\mathbf{b}_1, \mathbf{A}+\mathbf{b}_1-\mathbf{b}_3) + \Delta^2(\mathbf{A}+\mathbf{b}_1, \mathbf{A}+\mathbf{b}_1-\mathbf{b}_2) + \omega.$$

Complicated, but one can use a program to generate all lattice points !!

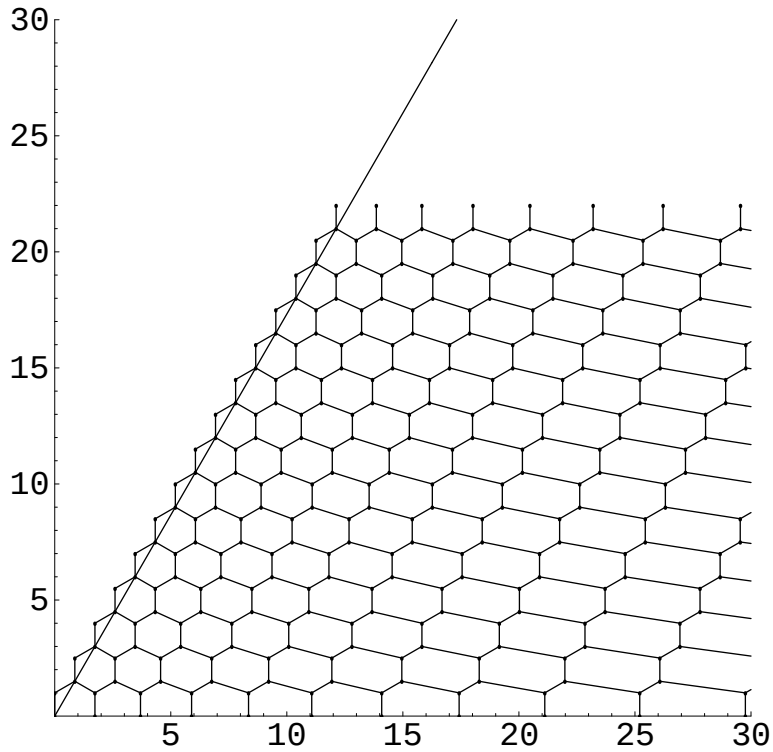
# Deformed lattices

Deformations



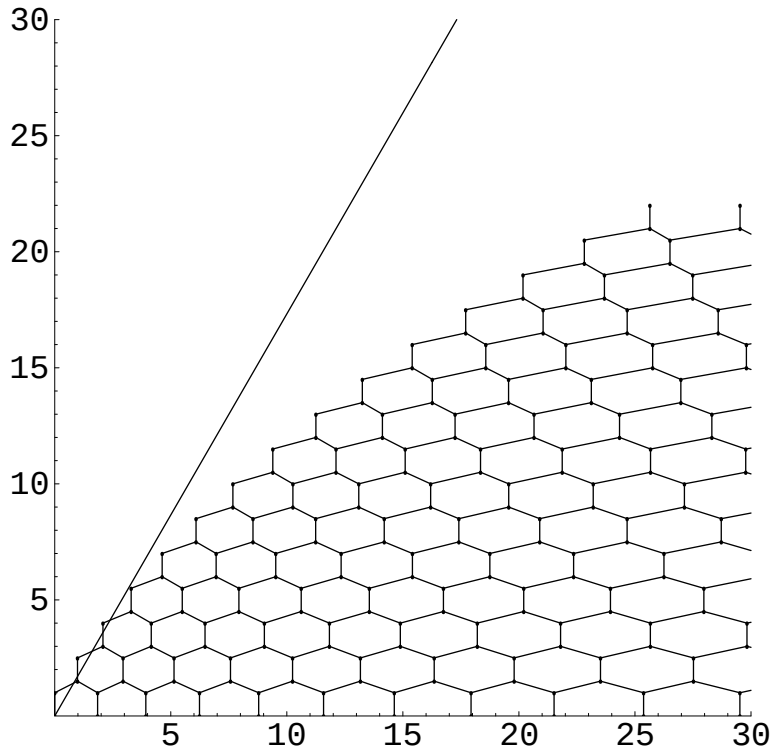
# Deformed lattices

Lattices produced with our recurrence relation. A regular hexagonal cell is used as a seed. A choice of deformation angle may produce periodicity in one direction (trivial case)



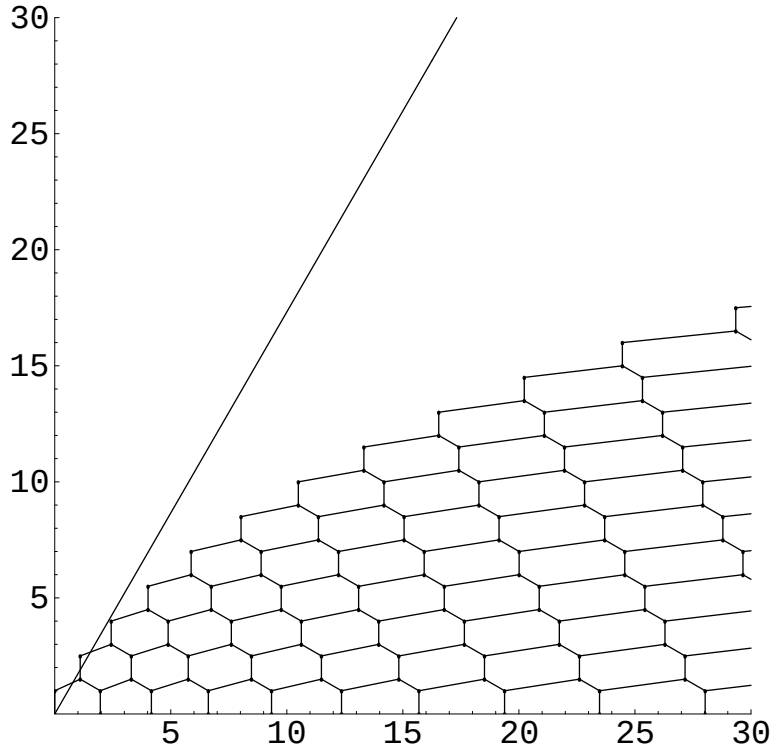
# Deformed lattices

Lattices produced with our recurrence relation. A regular hexagonal cell is used as a seed. No periodicity.



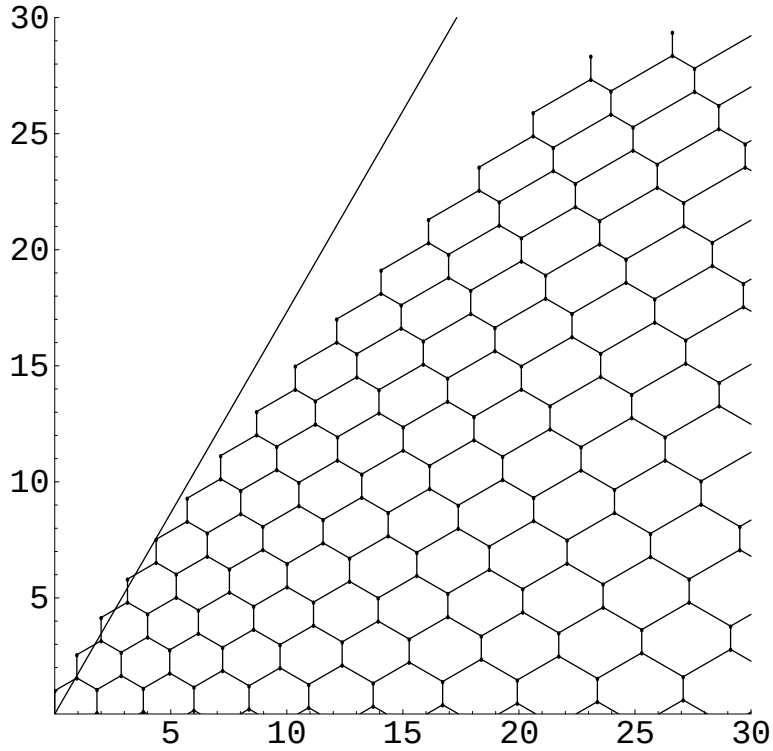
# Deformed lattices

Lattices produced with our recurrence relation. A regular hexagonal cell is used as a seed. No periodicity.



# Deformed lattices

Lattices produced with our recurrence relation. A regular hexagonal cell is used as a seed. No periodicity.



# Final result

The resulting hamiltonian of this problem is

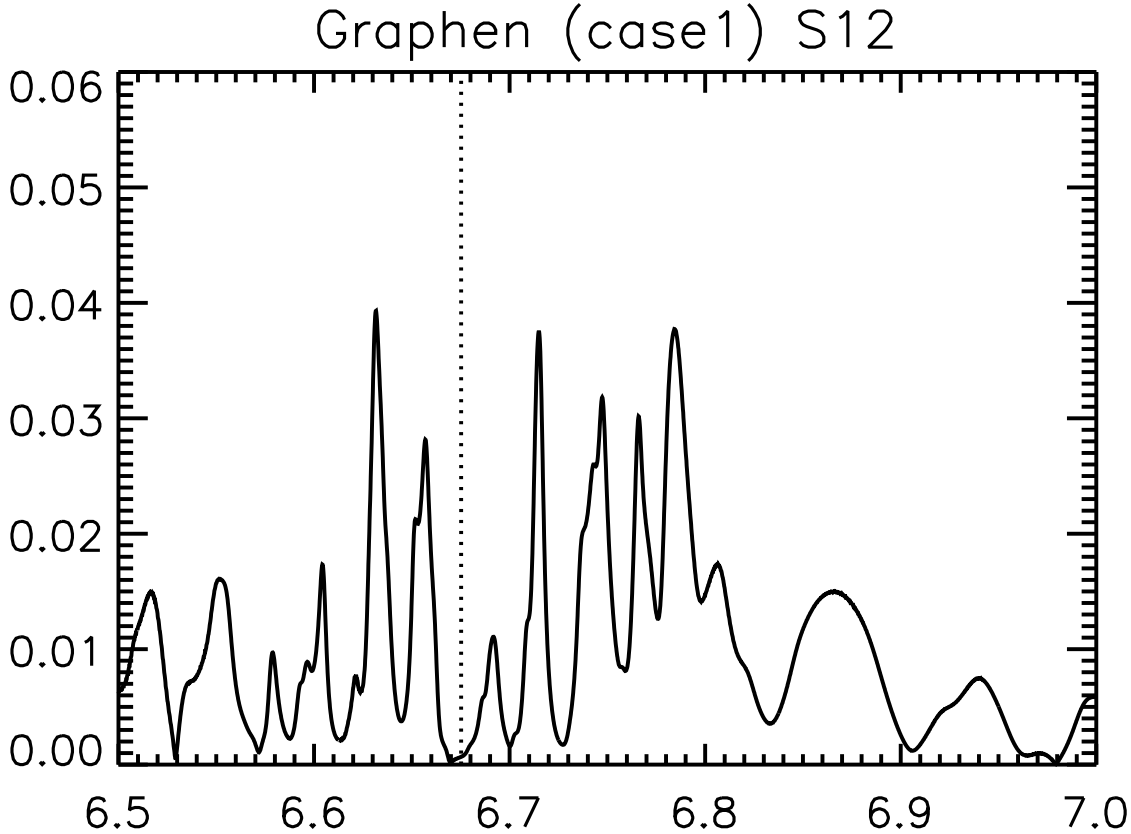
$$H = E_0 + \sigma_3 M + \sigma_+ a_R + \sigma_- a_R^\dagger \tag{30}$$

with eigenvalues

$$E(N_R) = E_0 \pm \sqrt{\omega(N_R + 1) + M^2}, \qquad 0 < N_R < \Delta^2/\omega \tag{31}$$

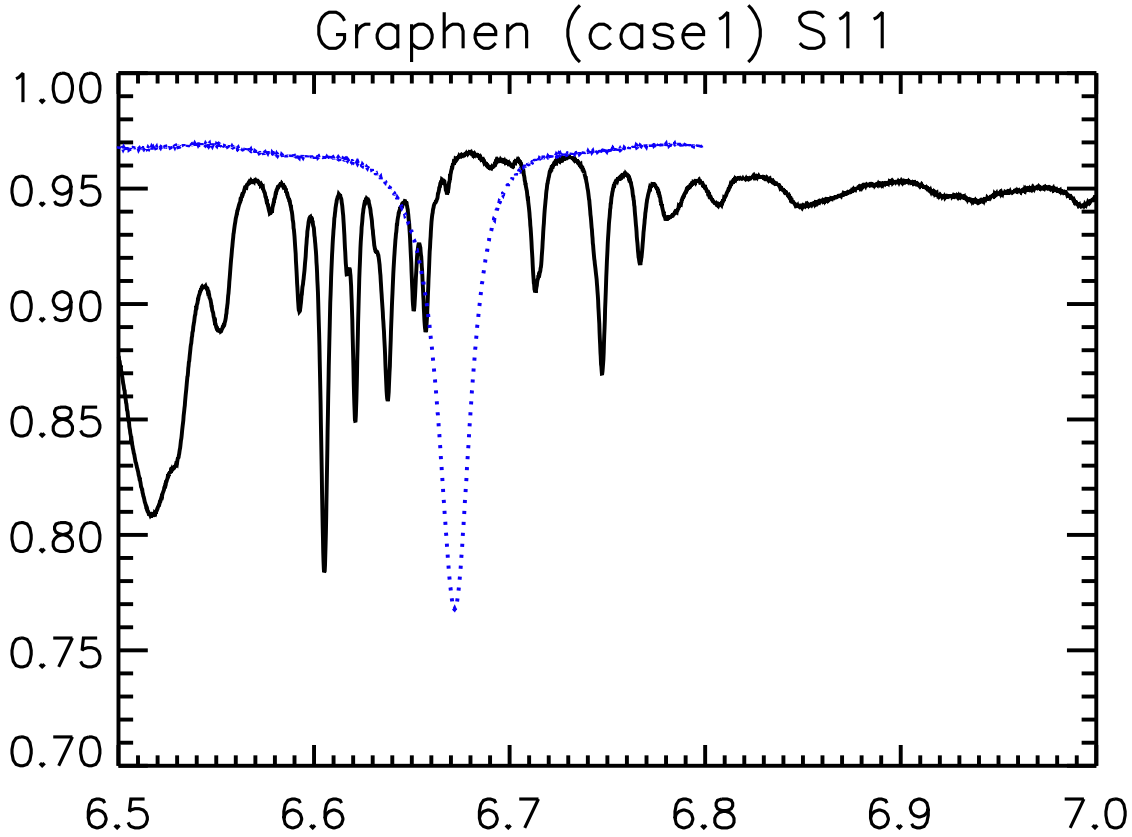
# Reflection

Preliminary experimental results. The blue line indicates the Dirac point. The equally spaced spectrum appears due to the deformation. The gap indicates the zero point energy of the oscillator



# Transmission

The equally spaced spectrum appears due to the deformation. The gap indicates the zero point energy of the oscillator



# The importance of tight binding

We claim that rotational symmetry around degeneracy points is a direct consequence of the tight binding approximation, as we shall see. It is well known that rotational symmetry in the Dirac equation demands a transformation of both orbital and spinorial degrees of freedom. It is in the orbital part that we shall concentrate by studying the energy surfaces around degeneracy points beyond the tight binding model. In our study, it will suffice to look inside the first Brillouin zone since the rest of the reciprocal lattice can be obtained by periodicity. Small deviations from degeneracy points (denoted by $\mathbf{k}_0$) in the form $\mathbf{k} = \mathbf{k}_0 + \boldsymbol{\kappa}$ give the energy

$$E = \Delta|\sum_i \exp\left(i\lambda(\mathbf{k}_0 + \boldsymbol{\kappa})\cdot\mathbf{b}_i\right)| \simeq \Delta\lambda|\boldsymbol{\kappa}|, \tag{32}$$

which is rotationally invariant in $\boldsymbol{\kappa}$.

A second-neighbor interaction of strength $\Delta'$ modifies the kinetic operator $\Pi$ as

$$\Pi = \Delta \sum_{i=1,2,3} T_{\mathbf{b}_i} + \Delta' \sum_{i=1,2,3} T_{\mathbf{a}_i} + T_{-\mathbf{a}_i}\,, \tag{33}$$

where the vectors $\mathbf{a}_i$ have now appeared, connecting a point with its six second neighbors. The energy equation becomes

$$E = |\Delta \sum_i \exp\left(i\lambda \mathbf{k}\cdot\mathbf{b}_i\right) + \Delta' \sum_i 2\cos\left(\lambda\mathbf{k}\cdot\mathbf{a}_i\right)|. \tag{34}$$

We expect a deviation of degeneracy points $\mathbf{k}_0'$, for which $\mathbf{k} = \mathbf{k}_0' + \boldsymbol{\kappa}$. Upon linearization of the exponentials in $\boldsymbol{\kappa}$ we find the energy

$$E \simeq \sqrt{(\boldsymbol{\kappa} \cdot \mathbf{u})^2 + (\boldsymbol{\kappa} \cdot \mathbf{v})^2} \tag{35}$$

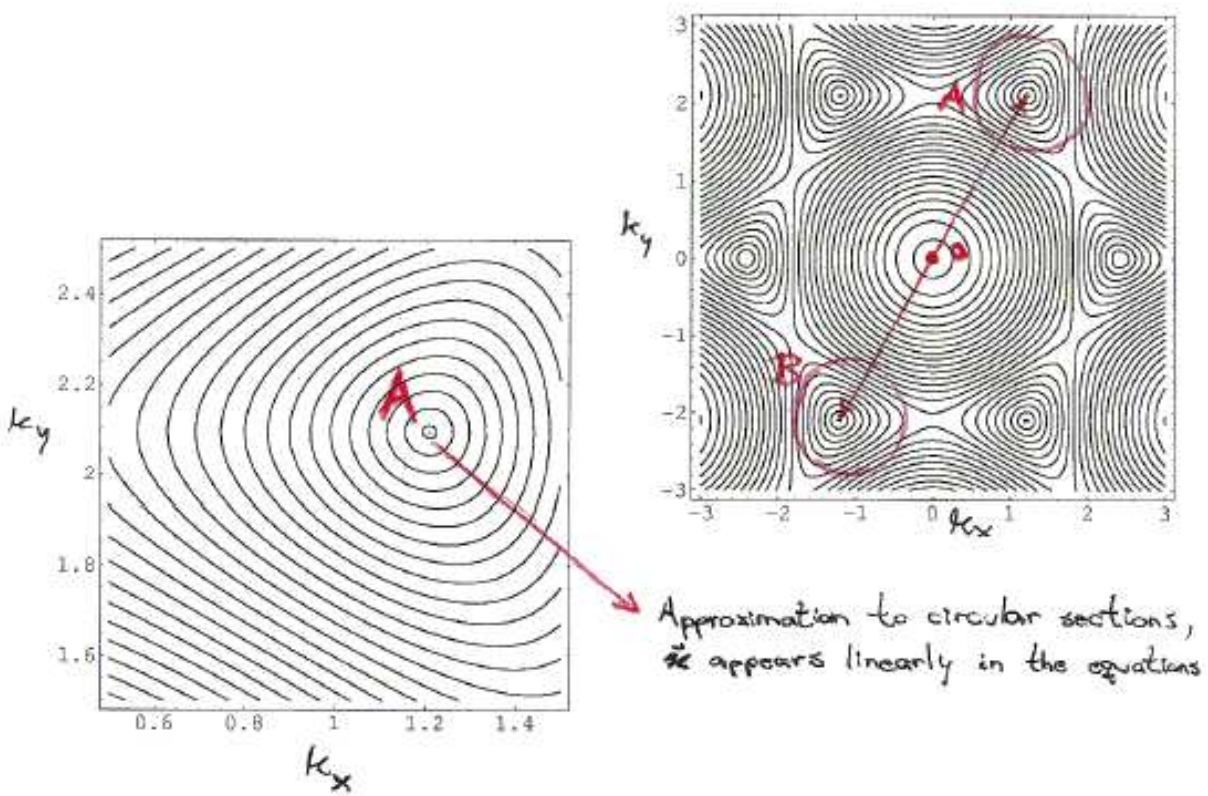
where the vectors are given by

$$\mathbf{u} = \lambda\Delta \sum_i \cos(\lambda \mathbf{k}_0' \cdot \mathbf{b}_i)\mathbf{b}_i \tag{36}$$

$$\mathbf{v} = \lambda\Delta \sum_i \sin(\lambda \mathbf{k}_0' \cdot \mathbf{b}_i)\mathbf{b}_i + 2\lambda\Delta' \sum_i \sin(\lambda \mathbf{k}_0' \cdot \mathbf{a}_i)\mathbf{a}_i \tag{37}$$

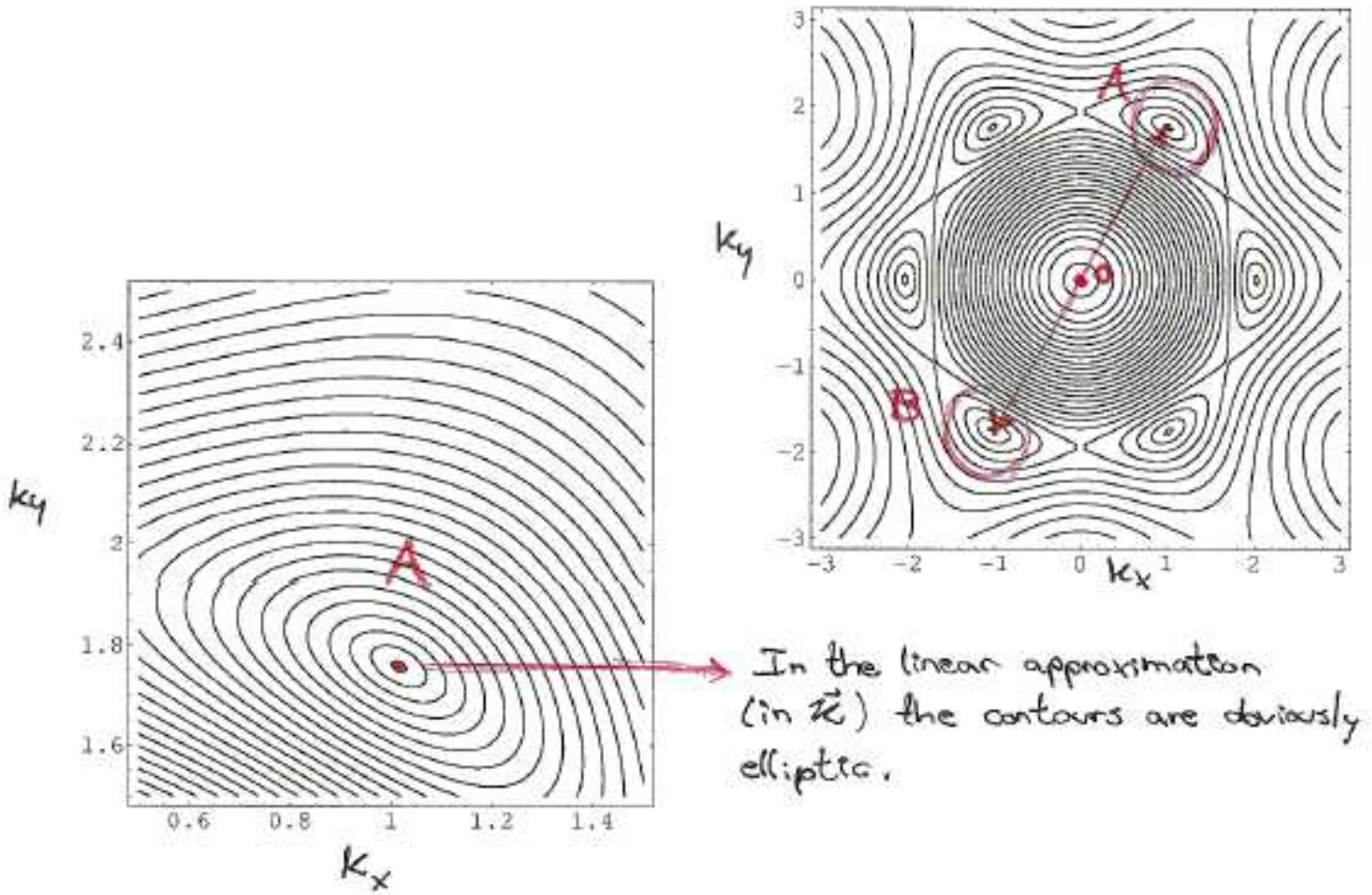
# Energy contours

First neighbour interaction, circular contours



# Energy contours

Second neighbour interaction, elliptic contours




- The presence of $\Delta'$ gives the energy surfaces (35) as cones with elliptic sections whenever $\boldsymbol{\kappa}$ is inside the first Brillouin zone. Regardless of how we complete the energy contours to recover periodicity, it is evident that the resulting surfaces are not invariant under rotations around degeneracy points. The circular case is recovered only when $\Delta' = 0$, leading to $\mathbf{k}'_0 = \mathbf{k}_0$. In this case, the vectors reduce to $\mathbf{v} = (1, 0), \mathbf{u} = (0, 1)$ when $\mathbf{k}_0$ is the degeneracy point at $(1/2\lambda, 0)$.
- In summary, extending the interactions to second neighbors has the effect of breaking the isotropy of space AROUND DEGENERACY POINTS, which is an essential property of the free Dirac theory.

# Conclusions

- We provide a useful description for a problem motivated by graphene and its emulation in electromagnetic billiards.
- Spectra, eigenfunctions and Dirac points have been reproduced.
- We have developed a method to analyze deformations through the algebraic properties of the system
- The experimental realization of this well known relativistic system is desirable.

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