AN ANALYTICAL TREATMENT FOR THREE NEUTRINO OSCILLATIONS IN THE EARTH

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SUMMARY: *The Magnus expansion for the evolution operator provides a convenient formalism to find approximate solutions to the problem of three neutrino oscillations in a medium with an arbitrarily varying density. This method allows us to incorporate in a simple way the Earth matter effects on the transition probabilities for neutrinos with a wide interval of energies, making possible an accurate description of such effects in the case of solar, accelerator, and atmospheric neutrinos.*

INTRODUCTION

Evolution operator of a system with Hamiltonian $\mathcal{H}$:

$$i\hbar\, \partial\mathcal{U}(t,t_0)/\partial t = \mathcal{H}\,\mathcal{U}(t,t_0),\quad \mathcal{U}(t_0,t_0)=I\,.$$

Magnus exponential expansion [1]:

$$\mathcal{U}(t,t_0)=\exp(\Omega(t,t_0)),\ \Omega=\sum_{n=1}^{\infty}\Omega_n$$

The first two terms of the series are:

$$\Omega_1(t,t_0)= -\tfrac{i}{\hbar}\int_{t_0}^{t} dt_1 H(t_1)\,,$$

$$\Omega_2(t,t_0)= -\tfrac{1}{2\hbar^2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 [H(t_1),H(t_2)]\,,$$

The Ω_n are anti-Hermitian, thus truncating the series at any order yields a unitary approximation for $\mathcal{U}(t,t_0)$.

Neutrino Mixing

PMNS mixing matrix: $U=\mathcal{O}_{23}\Gamma\mathcal{O}_{13}\Gamma^*\mathcal{O}_{12}$, with $\mathcal{O}_{ij}$ a rotation in the *ij* plane by the angle θ_{ij}, and the diagonal matrix $\Gamma=\mathrm{diag}(1,1,e^{i\delta})$ contains the CP phase δ.

$$\psi_\alpha(t)=\sum_{i=1}^{3}U_{\alpha i}\psi_i(t)\ ,\quad \psi_\alpha(t),\ \alpha=e,\mu,\tau\,.$$

Matter Evolution

In a medium the MSW Hamiltonian in the flavor basis is

$$\mathcal{H}(t)=U\,H_0\,U^\dagger+V(t)Y,\quad V(t)=\sqrt{2}G_F\,N_e(t)$$

$$H_0=\mathrm{diag}(0,\Delta_{21},\Delta_{31}), Y=\mathrm{diag}(1,0,0), \Delta_{ij}=\Delta m^2_{ij}/2E,$$

Diagonalization of $\mathcal{H}(t)$ defines the *adiabatic* basis $\{|\nu_{1m}(t)\rangle,|\nu_{2m}(t)\rangle,|\nu_{3m}(t)\rangle\}$, in which the evolution operator satisfies

$$i\hbar\frac{\partial}{\partial t}\mathcal{U}(t,t_0)=\left[\mathcal{H}_D(t)-i\hbar U_m^\dagger(t)\frac{\partial U_m(t)}{\partial t}\right]\mathcal{U}(t,t_0),$$

$$\mathcal{H}_D(t)=U_m(t)^\dagger\mathcal{H}(t)U_m(t)=\mathrm{diag}(E_1(t),E_2(t),E_3(t))$$

$U_m(t)$ is the mixing matrix in matter at time *t*.

$$\psi_\alpha(t)=\sum_{i=1}^{3}(U_m)_{\alpha i}(t)\psi_{i\,m}(t)$$

Parameters

$$\theta_{12}=34.4°,\ \theta_{13}=7.9°,\ \theta_{23}=42.3°,$$
$$\Delta m^2_{21}=7.67\times10^{-5}\mathrm{eV}^2,\ \Delta m^2_{31}=2.46\times10^{-3}\mathrm{eV}^2.$$

NEUTRINOS THROUGH THE EARTH

Mantle-Core-Mantle model, densities $V_{(M,C)}$ =(2.48, 5.95) $N_A\,\mathrm{cm}^{-3}$.

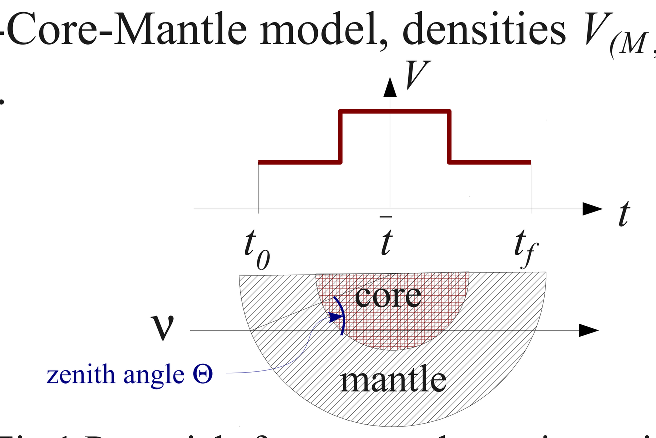

Fig.1 Potentials for a general neutrino trajectory.

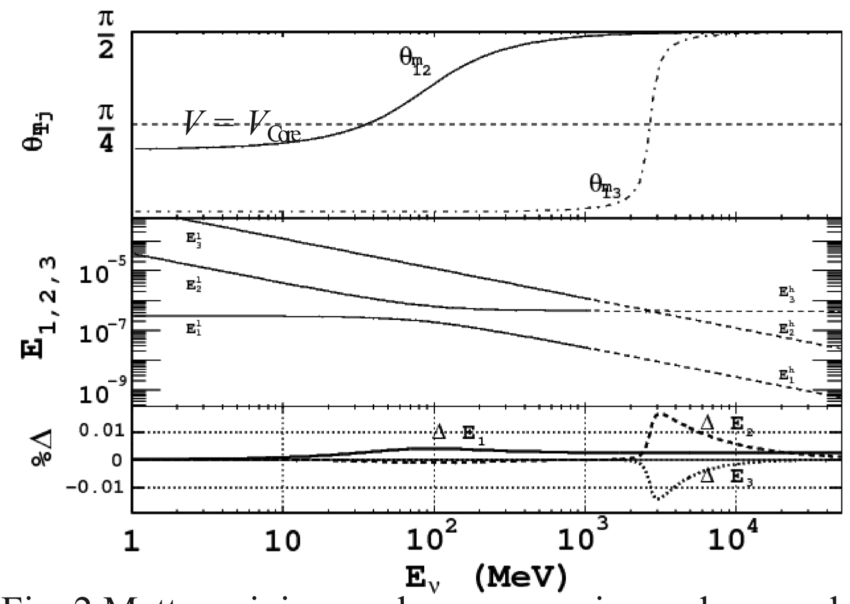

Fig. 2 Matter mixing angles, energy eigenvalues, and % residues *wrt.* exact *E*'s as a function of the E_ν, in the core.

APPROXIMATE DIAGONALIZATION

The smallness of $\Delta m^2_{21}/\Delta m^2_{31}\simeq 1/32$, defines two well separated regimes for propagation:

Low $E:\ V_{M,C}\simeq\Delta_{21}$, High $E:\ V_{M,C}\simeq\Delta_{31}$

$U_m(t)$ takes the approximate form:

$$U_m(t)\simeq\mathcal{O}_{23}\,\Gamma\,\mathcal{O}_{13}(\theta^m_{13}(t))\,\Gamma^*\,\mathcal{O}_{12}(\theta^m_{12}(t))$$

The matter angles $\theta^m_{12}(t)$ and $\theta^m_{13}(t)$ arise from diagonalizing $\mathcal{H}$ in each regime, where it reduces to that of a two-state problem. The approximate eigenvalues (Fig. 2) work well for all energies.

The transformation $\mathcal{U}(t,t_0)=\Gamma\ P(t,t_0)\ \mathcal{U}_P(t,t_0)\ \Gamma^*$, with $P(t,t_0)=\mathrm{diag}(e^{-i\alpha_1},e^{-i\alpha_2},e^{-i\alpha_3})$, and the phases $\alpha_k(t)=\frac{1}{\hbar}\int_{t_0}^{t}dt'E_k(t)$, removes $\mathcal{H}_D$ leading to

$$i\hbar\,\partial\mathcal{U}_P/\partial t=\mathcal{H}_P\,\mathcal{U}_P$$

where

$$\mathcal{H}_P(t)=\begin{bmatrix}0&h_a&0\\h_a^*&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&h_b\\0&0&h_c\\h_b^*&h_c^*&0\end{bmatrix}=\mathcal{H}_l(t)+\mathcal{H}_h(t),$$

$$h_a(t)=-i\hbar\,\dot\theta^m_{12}\ \exp-i[\alpha_2-\alpha_1],$$
$$h_b(t)=-i\hbar\,\dot\theta^m_{13}\ c^m_{12}\ \exp-i[\alpha_3-\alpha_1],$$
$$h_c(t)=-i\hbar\,\dot\theta^m_{13}\ s^m_{12}\ \exp-i[\alpha_3-\alpha_2],$$

FACTORIZED SOLUTION

Due to the separation of the two regimes

$$[\mathcal{U}_l(t,t_0),\mathcal{H}_h(t)]\simeq 0\ \ \forall t,$$
$$\Rightarrow\mathcal{U}_P(t_f,t_0)=\mathcal{U}_l(t_f,t_0)\times\mathcal{U}_h(t_f,t_0)$$

We solve for $\mathcal{U}_{l,h}(t,t_0)$ using the Magnus approximation up to 2nd order as in [2]. By taking $h_b(t)\approx 0$, we find

$$\mathcal{U}_P=\begin{bmatrix}u_{11}e^{-i\phi_{21}^{\bar t\to t_f}}&u_{12}e^{-i\phi_{21}^{\bar t\to t_f}}&0\\-u_{12}^*e^{i\phi_{21}^{\bar t\to t_f}}&u_{11}^*e^{i\phi_{21}^{\bar t\to t_f}}&0\\0&0&1\end{bmatrix}\times\begin{bmatrix}1&0&0\\0&v_{11}e^{-i\phi_{32}^{\bar t\to t_f}}&v_{12}e^{-i(\phi_{32}^{\bar t\to t_f}-\delta)}\\0&-v_{12}^*e^{i(\phi_{32}^{\bar t\to t_f}-\delta)}&v_{11}^*e^{i\phi_{32}^{\bar t\to t_f}}\end{bmatrix}$$

$$u_{11}=(\cos\xi^l-i\sin\xi^l\tfrac{\xi^l_{(2)}}{\xi^l})e^{i\phi_{21}^{\bar t\to t_f}},\ u_{12}=i\sin\xi^l\tfrac{\xi^l_{(1)}}{\xi^l}$$
$$v_{11}=(\cos\xi^h-i\sin\xi^h\tfrac{\xi^h_{(2)}}{\xi^h})e^{i\phi_{32}^{\bar t\to t_f}},\ v_{12}=i\sin\xi^h\tfrac{\xi^h_{(1)}}{\xi^h}e^{-i\delta}$$

with the phases $\phi_{ij}^{t_1\to t_2}=\int_{t_1}^{t_2}dt'(E_i(t')-E_j(t'))$,

and $\xi^{l,h}=\sqrt{(\xi^{l,h}_{(1)})^2+(\xi^{l,h}_{(2)})^2}$. Calculation of the $\xi^{l,h}$ factors exploits the symmetry of the potential (Fig. 1):

$$\xi_{(1)}=2\int_{\bar t}^{t_f}dt'\dot\theta^m_{ij}(t')\sin\phi_{ij}^{\bar t\to t'},$$
$$\xi_{(2)}=\int_{t_0}^{t_f}dt'\int_{t_0}^{t'}dt''\dot\theta^m_{ij}(t')\dot\theta^m_{ij}(t'')\sin\phi_{ij}^{t'\to t''}$$

Oscillation Probability $P(\nu_\mu\to\nu_e)$

Applying the evolution operator to a ν_μ entering the Earth we compute $P(\nu_\mu\to\nu_e)=|\psi_{\mu\to e}|^2$ with the amplitude

$$\begin{aligned}\psi_{\mu\to e}=\ &e^{-iI_1}\,U^m_{e1}(t_0)\Big[U^{m*}_{\mu1}(t_0)u_{11}+\big(U^{m*}_{\mu2}(t_0)v_{11}+U^{m*}_{\mu3}(t_0)v_{12}\big)\,u_{12}\,e^{-i\phi_{32}^{\bar t\to t_f}}\Big]\\&+e^{-iI_2}\,U^m_{e2}(t_0)\Big[-U^{m*}_{\mu1}(t_0)u^*_{12}\,e^{i\phi_{32}^{\bar t\to t_f}}+\big(U^{m*}_{\mu2}(t_0)v_{11}+U^{m*}_{\mu3}(t_0)v_{12}\big)\,u^*_{11}\Big]\\&+e^{-iI_3}\,U^m_{e3}(t_0)\big(-U^{m*}_{\mu2}(t_0)v^*_{12}+U^{m*}_{\mu3}(t_0)v^*_{11}\big)\end{aligned}$$

where $I_1=\alpha_1+\phi_{21}^{\bar t\to t_f}$, $I_2=\alpha_2+(\phi_{32}^{\bar t\to t_f}-\phi_{21}^{\bar t\to t_f})$, and $I_3=\alpha_3-\phi_{32}^{\bar t\to t_f}$.

Regeneration Factor

We can also calculate the regeneration factor

$$F_{reg}=P(\nu_2\to\nu_e)-P^{(0)}(\nu_2\to\nu_e)$$

evolving a ν_2 from t_0 to t_f, projecting it onto a ν_e, and subtracting the vacuum 2→*e* transition probability.

RESULTS WITH OPERATOR PRODUCT

$$\mathcal{U}_P(t_f,t_0)=\mathcal{U}_l(t_f,t_0)\times\mathcal{U}_h(t_f,t_0)$$

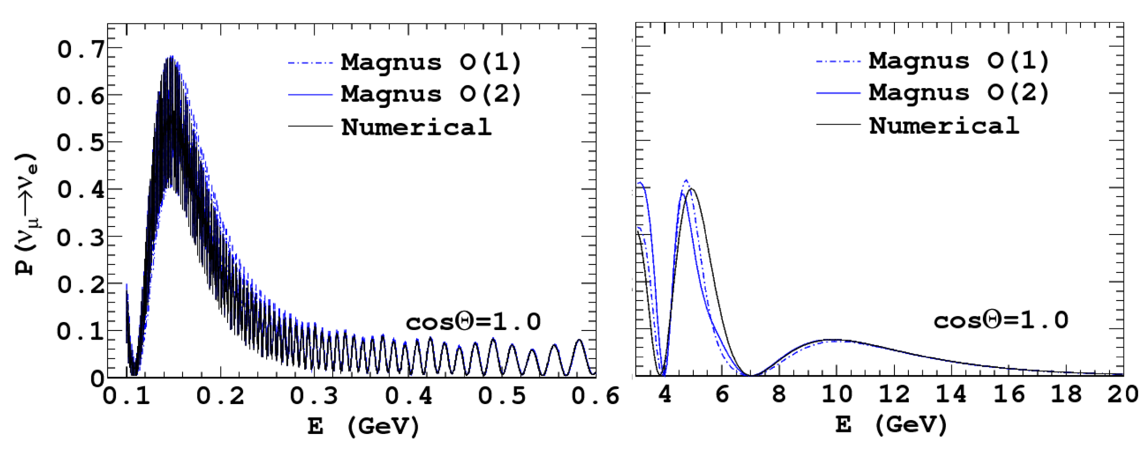

Fig. 3 Oscillation probability $P(\nu_\mu\to\nu_e)$ at low energies (left) and high energies (right).

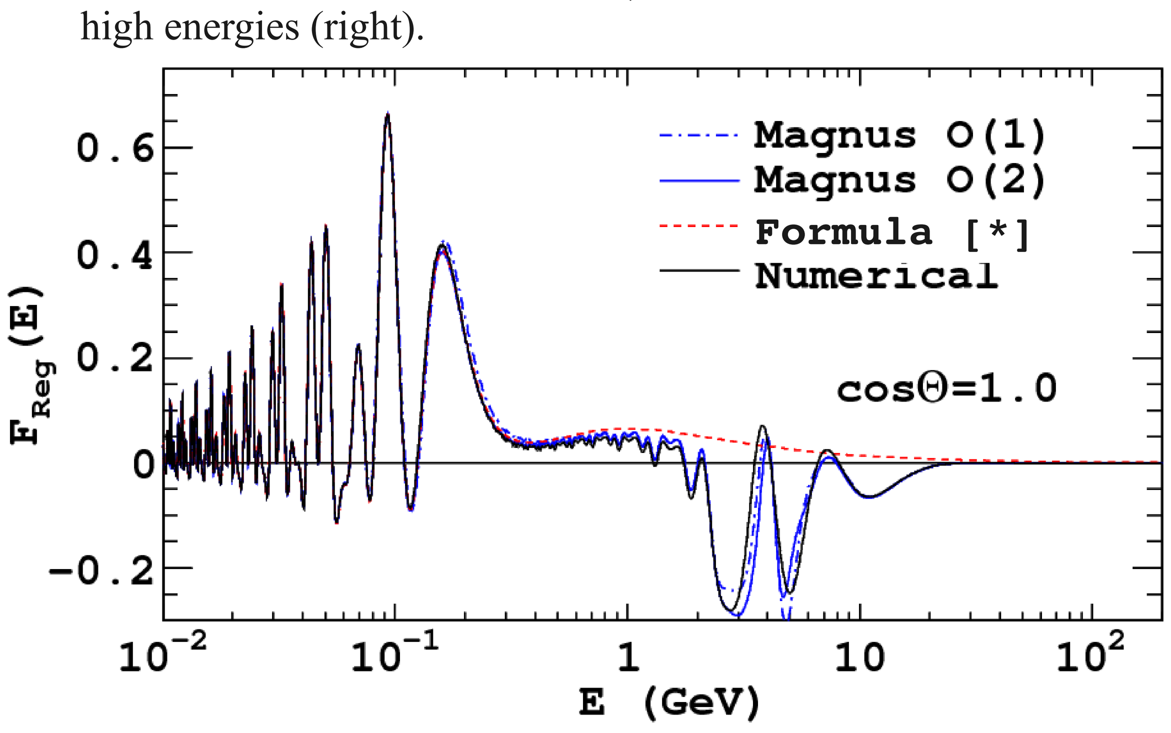

Fig. 4 Regeneration factor across a wide range of energies.

EFFECTIVE 3ν PROBLEM WITH MAGNUS

As an alternative approach we calculated F_{reg} for three neutrinos following Akhmedov *et al.* [3]:

$$F_{reg}=c^2_{13}\,F^{(2\nu)}_{reg}(E,\Delta m^2_{21},c^2_{13}V)\qquad[*]$$

calculating $F^{(2\nu)}_{reg}$ at low energies with a 2nd order Magnus approximation. The result is shown in Fig. 4 (red-dashed line) compared to the product approach, and in Fig. 5 (red-solid line).

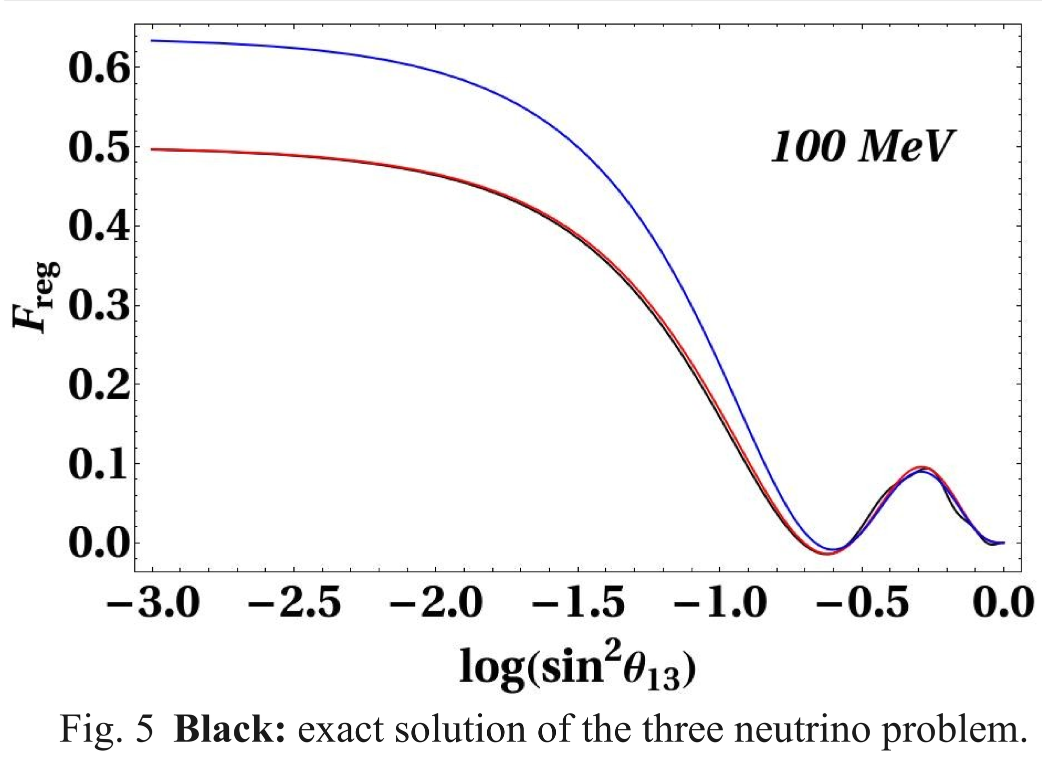

Fig. 5 **Black:** exact solution of the three neutrino problem.
Blue: perturbative solution (1st order) of the low energy two neutrino problem.
Red: Magnus solution up to 2nd order of the low energy two neutrino problem, including effect of the 3rd neutrino.

CONCLUSIONS

The Magnus expansion for the evolution operator implemented in the adiabatic basis gives an elegant and efficient formalism to describe three neutrino oscillations in a medium with varying density.

We have shown that the product of the solutions for the low and high energy regimes found with this method, renders simple semi-analytical formulas for the transition probabilities. When these formulas are applied to neutrinos traversing the Earth, they agree well with numerical calculations.

Finally, the Earth matter effects on low energy neutrinos can be accurately described by an effective three neutrino model where the two neutrino component is calculated using a 2nd order Magnus approximation.

Acknowledgements

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References

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[3] E. Akhmedov *et al.*, JHEP05 (2004) 057