CHAPTER 8

PATH INTEGRALS

Our discussion of time evolution has emphasized the importance of the Hamiltonian as the generator of time translations. In the 1940s R. P. Feynman discovered a way to express quantum dynamics in terms of the Lagrangian instead of the Hamiltonian. His path-integral formulation of quantum mechanics provides us with a great deal of insight into quantum dynamics, which alone makes it worthy of study. The computational complexity of using this formulation for most problems in nonrelativistic quantum mechanics is sufficiently high, however, that the path-integral method remained something of a curiosity until more recently, when it was realized that it also provides an excellent approach to quantizing a relativistic system with an infinite number of degrees of freedom, a quantum field.

8.1 THE MULTISLIT, MULTISCREEN EXPERIMENT

We can get the spirit of the path-integral approach to quantum mechanics by considering a straightforward extension of the double-slit experiment. Recall that the interference pattern in the double-slit experiment, shown in Fig. 8.1, can be understood as a probability distribution with the probability density at a point on the detecting screen arising from the superposition of two amplitudes, one for the particle to reach the point going through one of the slits and the other for the particle to reach the point going through the other slit. Suppose we increase the number of slits from two to three. Then there will be three amplitudes (see Fig. 8.2*a*) that we must add together to determine the probability amplitude that the particle reaches a particular point on the detecting screen. Suppose we next insert another opaque screen with two slits behind the initial screen (Fig. 8.2*b*).

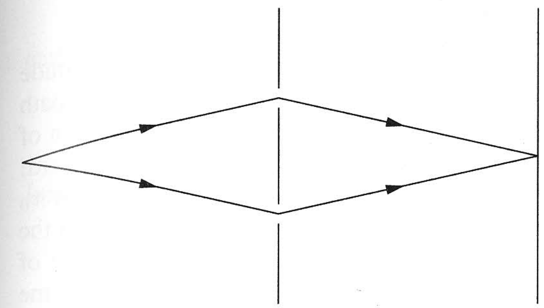
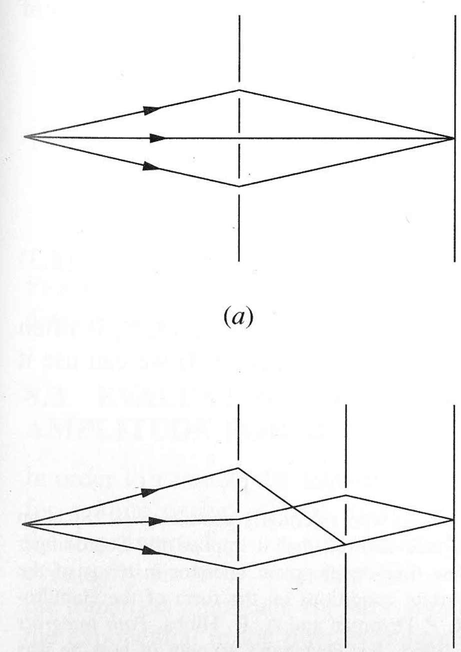


FIGURE 8.1
The two paths in the double-slit experiment. The amplitudes for these paths add together to produce an interference pattern on a distant detecting screen.

Now there are six possible paths that the particle can take to reach a point on the detecting screen; thus we must add six amplitudes together to obtain the total amplitude. One can imagine filling up the space between the source and the detecting screen with an infinite series of opaque screens and then eliminating these screens with an infinite number of slits in each screen. In this way, we see that the probability amplitude for the particle to arrive at a point on the detecting screen with no barriers in between the source and the detector must be the sum of the amplitudes for the particle to take every path between the source and the detection point.



(*a*)

(*b*)

FIGURE 8.2
(*a*) The three paths for a triple-slit experiment. (*b*) Three of the six paths that a particle may follow to reach a particular point on the detecting screen when an additional screen with two slits is inserted.

8.2 THE TRANSITION AMPLITUDE

We are now ready to see how we use quantum mechanics to evaluate the amplitude to take a particular path and how we add these amplitudes together to form a path integral.[1] In this chapter we will concentrate on a one-dimensional formulation of the path-integral formalism. The extension to three dimensions is straightforward.

We start with the amplitude $\langle x', t'|x_0, t_0\rangle$ for a particle that is at position x_0 at time t_0 to be at the position x' at time t'. In Chapter 4 when we introduced the subject of time evolution, we chose to set our clocks so that the initial state of the particle was specified at $t = 0$ and then considered the evolution for a time t. Here we are calling the initial time t_0 and considering the evolution for a time interval $t' - t_0$. Thus the transition amplitude is given by

$$\langle x', t'|x_0, t_0\rangle = \langle x'|\hat{U}(t' - t_0)|x_0\rangle = \langle x'|e^{-i\hat{H}(t'-t_0)/\hbar}|x_0\rangle \tag{8.1}$$

where $\hat{U}(t' - t_0)$ is the usual time-evolution operator and the Hamiltonian, which is assumed to be time-independent, is in general a function of the position and momentum operators: $\hat{H} = \hat{H}(\hat{p}_x, \hat{x})$. Of course, in the usual one-dimensional case

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x}) \tag{8.2}$$

Once we know the amplitude (8.1), we can use it to determine how any state $|\psi\rangle$ evolves with time, since we can write the state $|\psi\rangle$ as a superposition of position eigenstates:

$$\begin{aligned}\langle x'|\psi(t')\rangle &= \langle x'|e^{-i\hat{H}(t'-t_0)/\hbar}|\psi(t_0)\rangle \\ &= \int_{-\infty}^{\infty} dx_0\, \langle x'|e^{-i\hat{H}(t'-t_0)/\hbar}|x_0\rangle\langle x_0|\psi(t_0)\rangle \\ &= \int_{-\infty}^{\infty} dx_0\, \langle x', t'|x_0, t_0\rangle\langle x_0|\psi(t_0)\rangle \end{aligned}\tag{8.3}$$

The amplitude $\langle x', t'|x_0, t_0\rangle$, which appears within the integral in (8.3), is often referred to in wave mechanics as the propagator; according to (8.3) we can use it to determine how an arbitrary state propagates in time.

[1] Our approach is not that initially followed by Feynman, who essentially postulated (8.28) in an *independent* formulation of quantum mechanics and then showed that it implied the Schrödinger equation. Here, we start with the known form for the time-development operator in terms of the Hamiltonian and from it derive (8.28), subject to certain conditions on the form of the Hamiltonian. For a discussion of Feynman's approach, see R. P. Feynman and A. R. Hibbs, *Path Integrals and Quantum Mechanics*, McGraw-Hill, New York, 1965. For Feynman's account of how he was influenced by Dirac's work on this subject, see *Nobel Lectures—Physics*, vol. III, Elsevier Publication, New York, 1972. For a very nice physical introduction to path integrals, see R. P. Feynman, *QED—The Strange Theory of Light and Matter*, Princeton University Press, Princeton, N. J., 1985.

As an example, let's evaluate the propagator for a free particle using our earlier formalism. The Hamiltonian for a free particle is given by

$$\hat{H} = \frac{\hat{p}_x^2}{2m} \tag{8.4}$$

Inserting a complete set of momentum states

$$\int_{-\infty}^{\infty} dp\, |p\rangle\langle p| = 1 \tag{8.5}$$

in (8.1), we obtain

$$\begin{aligned}\langle x', t'|x_0, t_0\rangle &= \int_{-\infty}^{\infty} dp\, \langle x'|e^{-i\hat{p}_x^2(t'-t_0)/2m\hbar}|p\rangle\langle p|x_0\rangle \\ &= \int_{-\infty}^{\infty} dp\, \langle x'|p\rangle\langle p|x_0\rangle e^{-ip^2(t'-t_0)/2m\hbar}\end{aligned} \tag{8.6}$$

Using

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \tag{8.7}$$

we see that

$$\langle x', t'|x_0, t_0\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp\, e^{ip(x'-x_0)/\hbar} e^{-ip^2(t'-t_0)/2m\hbar} \tag{8.8}$$

This is a Gaussian integral, which can be evaluated using (D.7):

$$\langle x', t'|x_0, t_0\rangle = \sqrt{\frac{m}{2\pi\hbar i(t' - t_0)}}\, e^{im(x'-x_0)^2/2\hbar(t'-t_0)} \tag{8.9}$$

Problem 8.1 illustrates how we can use this expression for the propagator to determine how a Gaussian wave packet for a free particle evolves in time.

8.3 EVALUATING THE TRANSITION AMPLITUDE FOR SHORT TIME INTERVALS

In order to evaluate the transition amplitude $\langle x', t'|x_0, t_0\rangle$ for the interacting case for a finite period of time using the path-integral formalism, we first break up the time interval $t' - t_0$ into N intervals, each of size $\Delta t = (t' - t_0)/N$. We will eventually let $N \to \infty$ so that $\Delta t \to 0$. Thus we are interested first in evaluating the transition amplitude for very small time intervals. In this limit we can expand the exponential in the time-evolution operator in a Taylor series:

$$e^{-i\hat{H}\Delta t/\hbar} = 1 - \frac{i}{\hbar}\hat{H}(\hat{p}_x, \hat{x})\Delta t + O(\Delta t^2) \tag{8.10}$$

where the expression $O(\Delta t^2)$ includes the Δt^2 and higher powers of Δt terms. If we now evaluate the amplitude for a particle at x to be at x' a time Δt later, we obtain

$$\begin{aligned}\langle x'|e^{-i\hat{H}\Delta t/\hbar}|x\rangle &= \langle x'|\left[1 - \frac{i}{\hbar}\hat{H}(\hat{p}_x, \hat{x})\Delta t\right]|x\rangle + O(\Delta t^2) \\ &= \langle x'|\left[1 - \frac{i}{\hbar}\left(\frac{\hat{p}_x^2}{2m} + V(\hat{x})\right)\Delta t\right]|x\rangle + O(\Delta t^2) \qquad (8.11)\end{aligned}$$

It is easy to evaluate the action of $V(\hat{x})$ since the ket in (8.11) is an eigenstate of the position operator and therefore

$$V(\hat{x})|x\rangle = V(x)|x\rangle \qquad (8.12)$$

In order to evaluate the action of the kinetic energy operator, it is convenient to insert the complete set of momentum states (8.5) between the bra vector and the operator in (8.11) and then take advantage of

$$\langle p|\hat{p}_x = \langle p|p \qquad (8.13)$$

In this way we obtain

$$\begin{aligned}\langle x'|e^{-i\hat{H}\Delta t/\hbar}|x\rangle &= \int_{-\infty}^{\infty} dp\,\langle x'|p\rangle\langle p|\left[1 - \frac{i}{\hbar}\left(\frac{p^2}{2m} + V(x)\right)\Delta t\right]|x\rangle + O(\Delta t^2) \\ &= \int_{-\infty}^{\infty} dp\,\langle x'|p\rangle\langle p|\left[|1 - \frac{i}{\hbar}E(p,x)\Delta t\right]|x\rangle + O(\Delta t^2) \qquad (8.14)\end{aligned}$$

where

$$E(p,x) = \frac{p^2}{2m} + V(x) \qquad (8.15)$$

We now take advantage of (8.10) in reverse to write

$$1 - \frac{i}{\hbar}E(p,x)\Delta t = e^{-iE(p,x)\Delta t/\hbar} + O(\Delta t^2) \qquad (8.16)$$

Thus the transition amplitude (8.14) becomes

$$\begin{aligned}\langle x'|e^{-i\hat{H}\Delta t/\hbar}|x\rangle &= \int_{-\infty}^{\infty} dp\,\langle x'|p\rangle\langle p|x\rangle e^{-iE(p,x)\Delta t/\hbar} + O(\Delta t^2) \\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp\, e^{ip(x'-x)/\hbar}e^{-iE(p,x)\Delta t/\hbar} + O(\Delta t^2) \qquad (8.17)\end{aligned}$$

or simply

$$\langle x'|e^{-i\hat{H}\Delta t/\hbar}|x\rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\exp\left\{\frac{i}{\hbar}\left[p\frac{(x'-x)}{\Delta t} - E(p,x)\right]\Delta t\right\} + O(\Delta t^2) \qquad (8.18)$$

Equation (8.18) is deceptively simple in appearance. Although we characterized (8.16) as (8.10) in reverse, the exponential (8.10) contains the Hamiltonian *operator*, while the exponential (8.16) involves no operators at all. Where have the operators gone? The answer is that we have avoided much of the complexity of having to deal with the exponential of an operator by retaining just the terms through first order in Δt in (8.11). These complications are absorbed in the $O(\Delta t^2)$ term in (8.14). For example, if we were to try to calculate the Δt^2 term in (8.14), we would see that the fact that the position and momentum operators in the Hamiltonian do not commute prevents our replacing both these operators with ordinary numbers by inserting just a *single* complete set of momentum states. But if we consider the limit of the transition amplitude (8.18) as $\Delta t \to 0$, we can ignore these $O(\Delta t^2)$ complications. We will next see, however, that there is a penalty to pay for formulating quantum mechanics in a way that eliminates the operators that have been characteristic of our treatment of time development using the Hamiltonian formalism.

8.4 THE PATH INTEGRAL

We are now ready to evaluate the transition amplitude $\langle x', t'|x_0, t_0\rangle$ for a finite time interval. As we suggested earlier, we break up the interval $t' - t_0$ into N equal-time intervals Δt with intermediate times $t_1, t_2, \ldots, t_{N-1}$, as shown in Fig. 8.3. Therefore

$$\langle x', t'|x_0, t_0\rangle = \langle x'|\underbrace{e^{-i\hat{H}\Delta t/\hbar}\ldots e^{-i\hat{H}\Delta t/\hbar}e^{-i\hat{H}\Delta t/\hbar}}_{N\ \text{times}}|x_0\rangle \qquad (8.19)$$

We next insert complete sets of position states

$$\int_{-\infty}^{\infty} dx_i\,|x_i\rangle\langle x_i| = 1 \qquad i = 1, 2, \ldots, N-1 \qquad (8.20)$$

between each of these individual time-evolution operators:

$$\begin{aligned}&\langle x', t'|x_0, t_0\rangle = \\ &\int dx_1 \ldots \int dx_{N-1}\langle x'|e^{-i\hat{H}\Delta t/\hbar}|x_{N-1}\rangle\langle x_{N-1}|e^{-i\hat{H}\Delta t/\hbar}|x_{N-2}\rangle\ldots \\ &\times\langle x_2|e^{-i\hat{H}\Delta t/\hbar}|x_1\rangle\langle x_1|e^{-i\hat{H}\Delta t/\hbar}|x_0\rangle \qquad (8.21)\end{aligned}$$

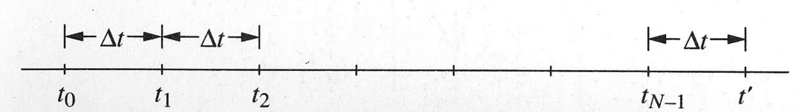


FIGURE 8.3
The interval $t' - t_0$ is broken into N time intervals, each of length Δt.

where each of the integrals is understood to run from $-\infty$ to ∞, as indicated in (8.20). Reading this equation from right to left, we see the amplitude for the particle at position x_0 at time t_0 to be at position x_1 at time $t_1 = t_0 + \Delta t$, multiplied by the amplitude for a particle at position x_1 at time $t_0 + \Delta t$ to be at position x_2 at time $t_2 = t_0 + 2\,\Delta t$. This sequence concludes with the amplitude for the particle to be at x' at time t' when it is at position x_{N-1} at a time Δt earlier. Figure 8.4 shows a typical path in the x-t plane for particular values of $x_1, x_2, \ldots, x_{N-1}$. Note that we are integrating over *all* values of $x_1, x_2, \ldots, x_{N-1}$ in (8.21). Thus, as we let $\Delta t \to 0$, we are effectively integrating over all paths that the particle can take in reaching the position x' at time t' when it starts at the position x_0 at time t_0.

We now use the expression (8.18) for the N amplitudes $\langle x_{i+1}|e^{-i\hat{H}\Delta t/\hbar}|x_i\rangle$ in (8.21), provided we are careful to insert the appropriate values for the initial and final positions in each case. If we let $N \to \infty$, and correspondingly $\Delta t \to 0$, we can ignore the $O(\Delta t^2)$ piece in each of the individual amplitudes, and the expression for the full transition amplitude is exactly given by

$$\langle x', t'|x_0, t_0\rangle = \lim_{N\to\infty} \int dx_1 \ldots \int dx_{N-1} \int \frac{dp_1}{2\pi\hbar} \cdots$$
$$\times \int \frac{dp_N}{2\pi\hbar} \exp\left\{\frac{i}{\hbar}\sum_{i=1}^{N}\left[p_i\frac{(x_i - x_{i-1})}{\Delta t} - E(p_i, x_{i-1})\right]\Delta t\right\} \tag{8.22}$$

where we have called the final position $x' = x_N$ in the exponent.

We now face a task that appears rather daunting: evaluating an infinite number of integrals. In fact, (8.22) involves both an infinite number of momentum and an infinite number of position integrals. Fortunately, for a Hamiltonian of the form (8.2), each of the momentum integrals is a Gaussian integral, which can

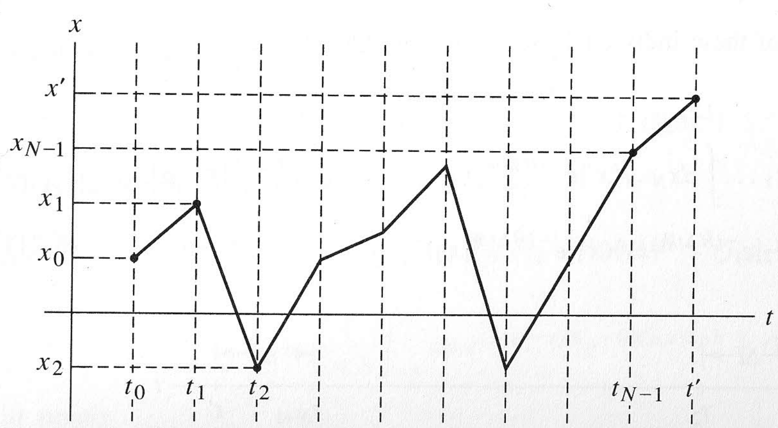


FIGURE 8.4
A possible path taken by the particle in going from position x_0 at time t_0 to a position x' at time t', with intermediate positions x_1 at time t_1, x_2 at time t_2, and so on.

be evaluated using (D.7) [with $a = i\Delta t/2m\hbar$ and $b = i(x_i - x_{i-1})/\hbar$]. A typical momentum integral is given by

$$\int \frac{dp_i}{2\pi\hbar}\exp\left[-i\frac{p_i^2\Delta t}{2m\hbar} + i\frac{p_i(x_i - x_{i-1})}{\hbar}\right]$$
$$= \sqrt{\frac{m}{2\pi\hbar i\Delta t}}\exp\left[\frac{i}{\hbar}\left(\frac{m\Delta t}{2}\right)\left(\frac{x_i - x_{i-1}}{\Delta t}\right)^2\right] \tag{8.23}$$

After doing all of the p integrals, we find

$$\langle x', t'|x_0, t_0\rangle = \lim_{N\to\infty}\int dx_1 \ldots \int dx_{N-1}\left(\frac{m}{2\pi\hbar i\Delta t}\right)^{N/2}$$
$$\times \exp\left\{\frac{i}{\hbar}\Delta t\sum_{i=1}^{N}\left[\frac{m}{2}\left(\frac{x_i - x_{i-1}}{\Delta t}\right)^2 - V(x_{i-1})\right]\right\} \tag{8.24}$$

Notice that as $N \to \infty$ and therefore $\Delta t \to 0$, the argument of the exponent becomes the standard definition of a Riemannian integral:

$$\lim_{N\to\infty,\,\Delta t\to 0}\frac{i}{\hbar}\Delta t\sum_{i=1}^{N}\left[\frac{m}{2}\left(\frac{x_i - x_{i-1}}{\Delta t}\right)^2 - V(x_{i-1})\right] = \frac{i}{\hbar}\int_{t_0}^{t'} dt\, L(x, \dot{x}) \tag{8.25}$$

where

$$L(x, \dot{x}) = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x) = \frac{1}{2}m\dot{x}^2 - V(x) \tag{8.26}$$

is the usual Lagrangian familiar from classical mechanics.[2]

Finally, it is convenient to express the remaining infinite number of position integrals using the shorthand notation

$$\int_{x_0}^{x'} \mathfrak{D}[x(t)] = \lim_{N\to\infty}\int dx_1 \ldots \int dx_{N-1}\left(\frac{m}{2\pi\hbar i\Delta t}\right)^{N/2} \tag{8.27}$$

which is a symbolic way of indicating that we are integrating over all paths connecting x_0 to x'. Then

$$\langle x', t'|x_0, t_0\rangle = \int_{x_0}^{x'} \mathfrak{D}[x(t)]\, e^{iS[x(t)]/\hbar} \tag{8.28}$$

where

$$S[x(t)] = \int_{t_0}^{t'} dt\, L(x, \dot{x}) \tag{8.29}$$

[2] If you are not familiar with the Lagrangian and the principle of least action, a brief but fun introduction is given in *The Feynman Lectures on Physics*, Vol. II, Chap. 19.

is the value of the action evaluated for a particular path taken by the particle. An integral such as (8.28) is referred to as a functional integral. In summing over all possible paths, we are really integrating over all possible functions $x(t)$ that meet the boundary conditions $x(t_0) = x_0$ and $x(t') = x'$.

Summarizing, in order to determine the amplitude for a particle at position x_0 at time t_0 to be found at position x' at time t', we consider *all* paths in the x-t plane connecting the two points. For each path $x(t)$, we evaluate the action $S[x(t)]$. Each path makes a contribution proportional to $e^{iS[x(t)]/\hbar}$, a factor that has unit modulus and depends on the path only through the phase factor $S[x(t)]/\hbar$. We then add up the contribution of each path. Note that in a formulation of quantum mechanics that starts with (8.28), operators need not be introduced at all. However, we must then face the issue of actually evaluating the path integral in order to determine the transition amplitude, or propagator. To give us some confidence that this is indeed feasible, at least in some cases, we first reconsider the evaluation of the transition amplitude (8.9) for a free particle, this time with the path-integral formalism. Then, in Section 8.6, we will use the path-integral formulation to examine the relationship between quantum and classical mechanics.

8.5 EVALUATION OF THE PATH INTEGRAL FOR A FREE PARTICLE

In order to evaluate the path integral (8.28) for a free particle, for which $V(x) = 0$, we retrace our derivation of (8.28) and break up the time interval $t' - t_0$ into N discrete Δt intervals:

$$\langle x', t'|x_0, t_0\rangle = \lim_{N\to\infty} \int dx_1 \ldots \times \int dx_{N-1} \left(\frac{m}{2\pi\hbar i\Delta t}\right)^{N/2} \exp\left[\frac{i}{\hbar}\Delta t \sum_{i=1}^{N} \frac{m}{2}\left(\frac{x_i - x_{i-1}}{\Delta t}\right)^2\right] \tag{8.30}$$

We introduce the dimensionless variables

$$y_i = x_i\sqrt{\frac{m}{2\hbar\Delta t}} \tag{8.31}$$

where again $x_N = x'$. Expressed in terms of these variables, the transition amplitude becomes

$$\langle x', t'|x_0, t_0\rangle = \lim_{N\to\infty} \left(\frac{m}{2\pi\hbar i\Delta t}\right)^{N/2} \left(\frac{2\hbar\Delta t}{m}\right)^{(N-1)/2} \int_{-\infty}^{\infty} dy_1 \ldots \times \int_{-\infty}^{\infty} dy_{N-1} \exp\left[i\sum_{i=1}^{N}(y_i - y_{i-1})^2\right] \tag{8.32}$$

Note that we have explicitly inserted the limits of integration.

Let's start with the y_1 integral, leaving aside for the moment the constants in front:

$$\int_{-\infty}^{\infty} dy_1\, e^{i[(y_2-y_1)^2+(y_1-y_0)^2]} = \sqrt{\frac{i\pi}{2}}\, e^{i(y_2-y_0)^2/2} \tag{8.33}$$

where we have taken advantage of (D.7). Fortunately, evaluating this integral has left us with another Gaussian. We are thus able to tackle the y_2 integral, again with the aid of (D.7):

$$\sqrt{\frac{i\pi}{2}}\int_{-\infty}^{\infty} dy_2\, e^{i(y_3-y_2)^2+(y_2-y_0)^2/2} = \sqrt{\frac{i\pi}{2}}\sqrt{\frac{i2\pi}{3}}\, e^{i(y_3-y_0)^2/3} = \sqrt{\frac{(i\pi)^2}{3}}\, e^{i(y_3-y_0)^2/3} \tag{8.34}$$

A comparison of the result of the y_1 integral (8.33) with the result of having done both the y_1 and the y_2 integrals in (8.34) suggests that the result of $(N-1)$ y integrals is just

$$\int_{-\infty}^{\infty} dy_1 \ldots \int_{-\infty}^{\infty} dy_{N-1} \exp\left[i\sum_{i=1}^{N}(y_i - y_{i-1})^2\right] = \sqrt{\frac{(i\pi)^{N-1}}{N}}\, e^{i(y_N-y_0)^2/N} \tag{8.35}$$

which can be established by induction. See Problem 8.2. Thus

$$\begin{aligned}\langle x', t'|x_0, t_0\rangle &= \lim_{N\to\infty}\left(\frac{m}{2\pi\hbar i\Delta t}\right)^{N/2}\left(\frac{2\hbar\Delta t}{m}\right)^{(N-1)/2}\left[\frac{(i\pi)^{N-1}}{N}\right]^{1/2} e^{i(y_N-y_0)^2/N} \\ &= \lim_{N\to\infty}\sqrt{\frac{m}{2\pi\hbar i N\Delta t}}\, e^{im(x_N-x_0)^2/2\hbar N\Delta t} \\ &= \sqrt{\frac{m}{2\pi\hbar i(t'-t_0)}}\, e^{im(x'-x_0)^2/2\hbar(t'-t_0)}\end{aligned} \tag{8.36}$$

where we have used $t' - t_0 = N\Delta t$ in the last step. This result is the same as we obtained with considerably less effort in Section 8.2 using the Hamiltonian formalism.

There is a limited class of problems with Lagrangians of the form

$$L(x, \dot{x}) = a + bx + cx^2 + d\dot{x} + ex\dot{x} + f\dot{x}^2 \tag{8.37}$$

where the integrals in the path-integral formulation are all Gaussian and the procedure we have outlined for the free particle can also be applied to determine the transition amplitude. In general, this is a fairly cumbersome procedure, but there are some shortcuts that can be used to determine the amplitude in these cases. The interested reader is urged to consult Feynman and Hibbs, *Path Integrals and Quantum Mechanics*. The main utility of the path-integral approach in nonrelativistic quantum mechanics is not, as you can probably believe, in explicitly determining the transition amplitude but in the alternative way it gives us of viewing time evolution in quantum mechanics and in the insight it gives us into the classical limit of quantum mechanics.

8.6 WHY SOME PARTICLES FOLLOW THE PATH OF LEAST ACTION

Equation (8.28) is an amazing result. Not only does *every* path contribute to the amplitude, but each path makes a contribution of the same magnitude. The only thing that varies from one path to the next is the value of the phase factor $S[x(t)]/\hbar$. Since quantum mechanics applies to all particles, why then, for example, does a macroscopic particle seem to follow a particular path at all?

In order to see which paths "count," let's consider an example. Suppose that at $t = 0$ a *free* particle of mass m is at the origin, $x = 0$, and that we are interested in the amplitude for the particle to be at $x = x'$ when $t = t'$. There are clearly an infinite number of possible paths between the initial and the final point. One such path, indicated in Fig. 8.5, is

$$x = \left(\frac{x'}{t'}\right)t \tag{8.38}$$

This is, of course, the path that a classical particle with no forces acting on it and moving at a constant speed $v = x'/t'$ would follow. For this path, $\dot{x} = x'/t'$, $L = m\dot{x}^2/2 = mx'^2/2t'^2$, and

$$S_{\text{cl}} = S[x_{\text{cl}}(t)] = \int_0^{t'} dt\, \frac{m}{2}\left(\frac{x'^2}{2t'^2}\right) = \frac{mx'^2}{2t'} \tag{8.39}$$

If we evaluate the phase $S_{\text{cl}}/\hbar$ for typical macroscopic parameters: $m = 1$ g, $x' = 1$ cm, and $t' = 1$ s, we find that the phase has the very large value of roughly $(1/2) \times 10^{27}$ radians.

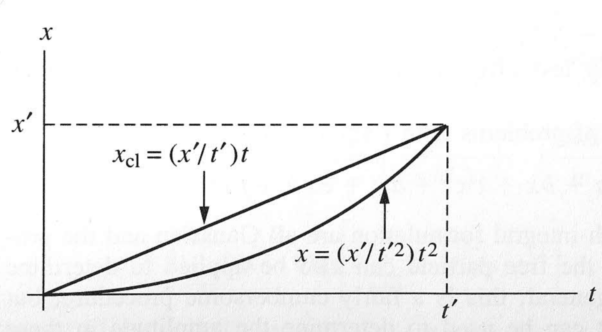


FIGURE 8.5
Two paths connecting the initial position $x = 0$ at $t = 0$ and the final position x' at time t': the classical path for a free particle, $x = (x'/t')t$, and the path $x = (x'/t'^2)t^2$.

We also choose another path, which is also depicted in Fig. 8.5, namely,

$$x = \left(\frac{x'}{t'^2}\right)t^2 \tag{8.40}$$

This path, which is characteristic of a particle undergoing uniform acceleration, is clearly *not* the classical path for a particle without any forces acting on it. For this path we find $L = m\dot{x}^2/2 = 2mx'^2t^2/t'^4$ and

$$S\left(x't^2/t'^2\right) = \int_0^{t'} dt\left(\frac{2mx'^2}{t'^4}\right)t^2 = \frac{2mx'^2}{3t'} \tag{8.41}$$

The value of the phase is roughly $(2/3) \times 10^{27}$ radians for the same macroscopic parameters.

Although the phases determined from (8.39) and (8.41) are different, what really distinguishes the classical path from any other is *not* the actual value of the action itself. Rather, the classical path is the path of least action, or, more precisely, the one for which the action is an extremum. To illustrate this explicitly, we consider a set of paths in the neighborhood of the two paths that we are using as examples. In the vicinity of the classical path, we take the set of paths

$$x = \frac{x'}{t'}\left[t + \varepsilon\frac{t(t-t')}{t'}\right] \tag{8.42}$$

where each value of the parameter ε labels a different path that deviates slightly from the classical path if ε is small. Notice that $x(t)$ still satisfies the initial and final conditions $x(0) = 0$ and $x(t') = x'$, respectively. It is straightforward to calculate the action:

$$S = \int_0^{t'} dt\,\frac{m\dot{x}^2}{2} = \int_0^{t'} dt\,\frac{m}{2}\left(\frac{x'}{t'}\right)^2\left[1 + \varepsilon\frac{(2t-t')}{t'}\right]^2$$
$$= \frac{m}{2}\frac{x'^2}{t'}\left(1 + \frac{\varepsilon^2}{3}\right) = S_{\text{cl}}\left(1 + \frac{\varepsilon^2}{3}\right) \tag{8.43}$$

The important thing to notice is that the change in the action depends on ε^2; there is no term linear in ε. The action is indeed a minimum; varying the path away from the classical path only increases the action from its value (8.39). Because the first-order contribution to the action vanishes:

$$\delta S = \left(\frac{\partial S}{\partial \varepsilon}\right)_{\varepsilon=0} \varepsilon = 0 \tag{8.44}$$

the contribution through first order of each of the paths to the path integral is proportional to

$$e^{i(S_{\text{cl}}+\delta S)/\hbar} = e^{iS_{\text{cl}}/\hbar} \tag{8.45}$$

Thus the amplitudes for the paths in the vicinity of the classical path will have roughly the same phase as does the classical path and will, therefore, add together constructively.

If, on the other hand, we consider the nonclassical path (8.40), we can also determine the action for a set of paths in its neighborhood,

$$x = \frac{x'}{t'^2}\left[t + \varepsilon\frac{t(t-t')}{t'}\right]^2 \tag{8.46}$$

which again satisfy $x(0) = 0$ and $x(t') = x'$. If we now calculate the action for (8.46), we obtain

$$S = \frac{2mx'^2}{3t'}\left(1 + \frac{\varepsilon}{2} + \cdots\right) = S\left(x't^2/t'^2\right)\left(1 + \frac{\varepsilon}{2} + \cdots\right) \tag{8.47}$$

Here, in agreement with the principle of least action, the first-order correction $\delta S = (\partial S/\partial\varepsilon)_{\varepsilon=0}\varepsilon \neq 0$. Some neighboring paths, in this case those with $\varepsilon < 0$, *reduce* the value of the action from its value for the path (8.40). The contribution through first order of the paths in the vicinity of the path (8.40) to the path integral is $e^{i(S+\delta S)/\hbar}$. Thus, in general, paths in the neighborhood of the nonclassical path may be out of phase with each other and may interfere destructively.

A useful pictorial way to show how this cancellation arises is in terms of phasors. For convenience, let's assume that we can label the paths discretely instead of continuously. When we add the complex numbers

$$e^{iS[x_1(t)]/\hbar} = \cos S[x_1(t)]/\hbar + i\sin S[x_1(t)]/\hbar \tag{8.48a}$$

and

$$e^{iS[x_2(t)]/\hbar} = \cos S[x_2(t)]/\hbar + i\sin S[x_2(t)]/\hbar \tag{8.48b}$$

together for two paths, we just add the real parts and the imaginary parts together separately. The magnitude of this complex number is of course given by

$$\left|e^{iS[x_1(t)]/\hbar} + e^{iS[x_2(t)]/\hbar}\right| \tag{8.49}$$

$$= \left(\{\cos S[x_1(t)]/\hbar + \cos S[x_2(t)]/\hbar\}^2 + \{\sin S[x_1(t)]\hbar + \sin S[x_2(t)]/\hbar\}^2\right)^{1/2}$$

We can recognize this as the same procedure we would use to find the length of an ordinary vector resulting from the addition of two vectors, $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$, namely,

$$|\mathbf{V}| = \left[(V_{1x} + V_{2x})^2 + (V_{1y} + V_{2y})^2\right]^{1/2} \tag{8.50}$$

Thus, if we indicate the complex amplitudes (8.48) by vectors in the complex plane, with the real part of the amplitude plotted along one axis and the imaginary part of the amplitude plotted along the other axis, the complex number resulting from the addition of the two amplitudes (8.48*a*) and (8.48*b*) is just the vector sum, as shown in Fig. 8.6.

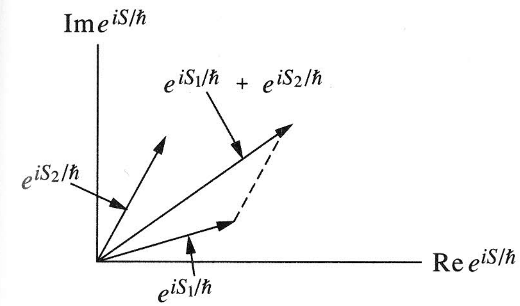


FIGURE 8.6
The addition of the amplitudes $e^{iS[x_1(t)]/\hbar}$ and $e^{iS[x_2(t)]/\hbar}$ is carried out using phasors. Each of the amplitudes is represented by an arrow of unit length in the complex plane, with an orientation angle, or phase angle, $S[x(t)]/\hbar$. The rule for adding the two amplitudes is the same as for ordinary vector addition.

What happens as we add up the contributions of the nonclassical path (8.40) and its neighbors? Notice that the first-order change in the action from (8.47) is proportional to the value S of the action itself multiplied by ε. As ε changes away from zero, the phase of the neighboring path changes. In the particular case (8.46), we see that when $\varepsilon S(x't^2/t'^2)/2\hbar = 2\pi$, the phase has returned to its initial value, modulo 2π. Thus if $S/\hbar$ is 10^{27} for some typical macroscopic parameters, ε need only reach the value $\varepsilon = 4\pi \times 10^{-27}$ to satisfy this condition. In Fig. 8.7*a* we add up the arrows for a discrete set of paths representing those between $\varepsilon = 0$ and $\varepsilon = 4\pi \times 10^{-27}$. These arrows form a closed "circle" and therefore sum to zero. Thus, the contribution from these paths cancel each other and hence do not contribute to the path integral (8.28). On the other hand, in the vicinity of the classical path (8.38), the first-order contribution to the action vanishes and thus the paths in the vicinity of the classical path have the same phase and add together constructively (Fig. 8.7*b*). This coherence will eventually break down, when the phase shift due to nearby paths reaches a value on the order of π. For our specific example (8.46), this means $S_{\text{cl}}\varepsilon^2/3\hbar \simeq \pi$, or $\varepsilon \simeq 10^{-13}$ for the macroscopic parameters. This is clearly a very tight constraint for a macroscopic particle, since the paths that count do not deviate far from the path of least action.

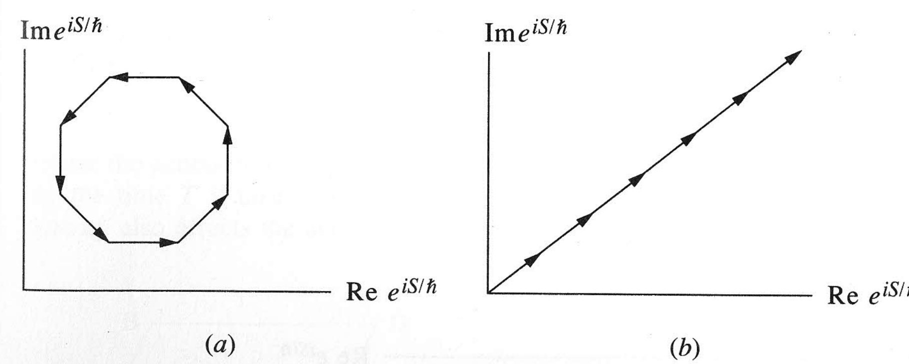


FIGURE 8.7
(*a*) The sum of a discrete set of amplitudes representing those in the vicinity of the nonclassical path. Since these arrows form a closed "circle," their sum vanishes. (*b*) In the vicinity of the classical path, the amplitudes, which all have the same phase to first order, sum to give a nonzero contribution to the path integral.

But the classical path is still important because *only* in its vicinity can many paths contribute to the path integral coherently. In the neighborhood of any other path, the contributions of neighboring paths cancel each other (see Fig. 8.8). Quantum mechanics thus allows us to understand how a particle knows to take the path of least action, at least in classical physics: the particle actually has an amplitude to take all paths.

Our numerical examples in this section so far have been entirely about a macroscopic particle. What happens if we replace the 1 g mass with an electron? Notice that the *phase difference* between the two paths in Fig. 8.5 is given by

$$\frac{\Delta S}{\hbar} = \frac{S(x'^2/t'^2) - S_{\text{cl}}}{\hbar} = \frac{mx'^2}{6t'\hbar} \tag{8.51}$$

While for $m = 1$ g with $x' = 1$ cm and $t' = 1$ s this phase difference is about $(1/6) \times 10^{27}$ radians, the phase difference between the two paths for the electron, for which $m \simeq 10^{-27}$ g, is only $\frac{1}{6}$ radian. Thus for an electron even the path $x = x't^2/t'^2$ is essentially coherent with the classical path $x = x't/t'$. Because there are many more paths that can contribute coherently to the path integral for the electron than there are for the macroscopic particle, the motion of the electron in this case should be extremely nonclassical in nature.

This last example with the electron is sufficient to cause us to wonder again about the double-slit experiment. Why do we see a clear interference pattern arising from the interference of the amplitudes to take just the two paths shown in Fig. 8.1? Why don't other paths, such as the one indicated in Fig. 8.9, contribute? The answer is that the action for the paths indicated in Fig. 8.1 is actually much larger than the previous example might lead you to think. For example, electrons with 50 eV of kinetic energy, a typical value for electron diffraction experiments, have a speed of 4×10^8 cm/s. Thus if we take $x' = 40$ cm as a typical size scale for the double-slit experiment and $t' = 10^{-7}$ s so that the speed has the proper

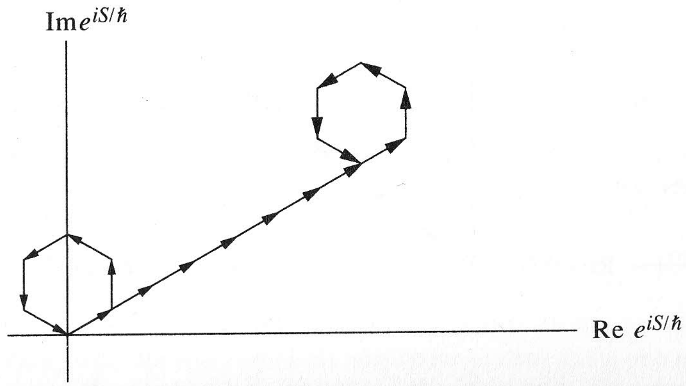


FIGURE 8.8
A schematic diagram using phasors showing for a macroscopic particle how the classical path *and* its neighbors dominate the path integral, while other paths give no net contribution as they and their neighbors cancel each other.

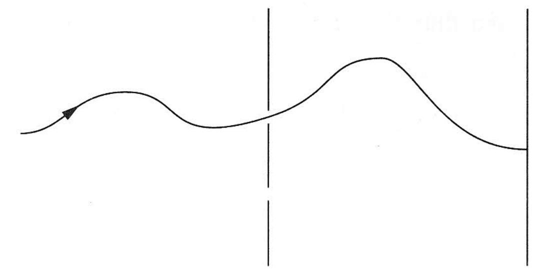


FIGURE 8.9
A path that does not contribute coherently to the double-slit experiment illustrated in Fig. 8.1.

magnitude, we find the phase $S/\hbar$ for the straight-line path in Fig. 8.5 to be 7×10^9. When the phase is this large, only a small deviation away from the classical path will cause coherence to be lost. The large size of the action is also the reason that we can use classical physics to aim an electron gun in a television set, where the electrons may have an energy of 5 KeV, or to describe the motion of atoms through the magnets in the Stern-Gerlach experiments in Chapter 1.

8.7 QUANTUM INTERFERENCE DUE TO GRAVITY

We now show how we can use path integrals to analyze a striking experiment illustrating the sensitivity of the neutron interferometer that we first introduced in Section 4.3. An essentially monochromatic beam of thermal neutrons is split by Bragg reflection by a perfect slab of silicon crystal at A. One of the beams follows path ABD and the other follows path ACD, as shown in Fig. 8.10. In general, there will be constructive or destructive interference at D depending on the path difference between these two paths. Suppose that the interferometer initially lies in a horizontal plane so that there are no gravitational effects. We then rotate the plane formed by the two paths by angle δ about the segment AC. The segment BD is now higher than the segment AC by $l_2 \sin\delta$. Thus there will be an additional gravitational potential energy $mgl_2 \sin\delta$ along this path, which alters the action and hence the amplitude to take the path BD by the factor

$$e^{-i(mgl_2 \sin\delta)T/\hbar} \tag{8.52}$$

where the action in the exponent is the negative of the potential energy multiplied by the time T it takes for the neutron to traverse the segment BD. Of course, gravity also affects the action in traversing the segment AB, but this phase shift

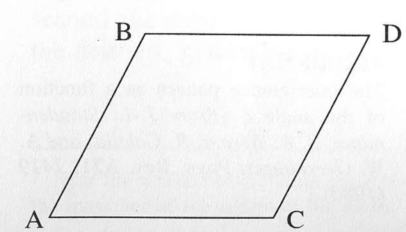


FIGURE 8.10
A schematic of the neutron interferometer. The interferometer, initially lying in a horizontal plane, can be rotated vertically about the axis AC by an angle δ.

is the same as for the segment CD, and thus the phase difference between the path ABD and the path ACD is given by

$$\frac{S[\text{ABD}] - S[\text{ACD}]}{\hbar} = -\frac{mgl_2T\sin\delta}{\hbar} = -\frac{m^2gl_2l_1\sin\delta}{\hbar p} = -\frac{m^2gl_2l_1\lambda\sin\delta}{2\pi\hbar^2} \tag{8.53}$$

where we have used the de Broglie relation $p = h/\lambda$ to express this phase difference in terms of the wavelength of the neutrons. Figure 8.11 shows the interference fringes that are produced as δ varies from $-45°$ (BD below AC) to $+45°$ (BD above AC) for neutrons with $\lambda = 1.419$ Å. The contrast of the interference pattern dies out with increasing angle of rotation because the interferometer bends and warps slightly (on the scale of angstroms) under its own weight as it is rotated about the axis AC.

Notice in the classical limit that as $\hbar \to 0$, the spacing between the fringes in (8.53) becomes so small that the interference pattern effectively washes out. This interference is, in fact, the only gravitational effect that depends in a nontrivial way on quantum mechanics that has so far been observed.[3] Now, not surprisingly, neutrons are observed to "fall" in a gravitational field,[4] but from (6.33) and (6.34) we see for a gravitational field pointing in the negative x direction that

$$\frac{d^2\langle x\rangle}{dt^2} = -g \tag{8.54}$$

[3] On a microscopic scale, where most quantum effects are observed, gravitation is an extremely weak force. For example, the ratio of the electromagnetic and the gravitational force between an electron and a proton is $Gm_em_p/e^2 = 4 \times 10^{-40}$.

[4] A. W. McReynolds, *Phys. Rev.* **83,** 172, 233 (1951); J. W. T. Dabbs, J. A. Harvey, D. Paya, and H. Horstmann, *Phys. Rev.* **139,** B756 (1965).

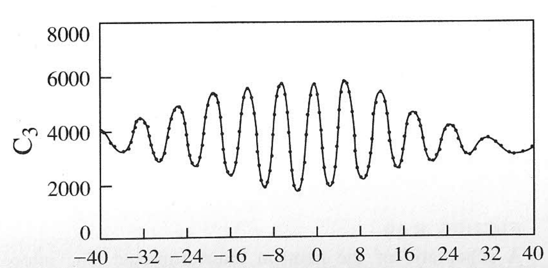


FIGURE 8.11
The interference pattern as a function of the angle δ [*From J.-L. Staudenmann, S. A. Werner, R. Colella, and A. W. Overhauser,* Phys. Rev. **A21,** *1419 (1980).*]

which does not depend on Planck's constant at all. Neither does (8.54) depend on value of the mass m. This lack of dependence on m is a consequence of the equivalence of inertial mass m_i, which would appear on the left-hand side of (8.54) as the m_ia of Newton's law, and the gravitational mass m_g, which appears in the right-hand side in the gravitational force.[5] All bodies fall at the same rate because of this equivalence. While this equivalence has been well tested in the classical regime, our result (8.53), which when expressed in terms of m_i and m_g becomes

$$\frac{S[\text{ABD}] - S[\text{ACD}]}{\hbar} = -\frac{m_im_ggl_2l_1\lambda\sin\delta}{2\pi\hbar^2} \tag{8.55}$$

provides us with a test of the equivalence between inertial and gravitational mass at the quantum level. The determination of m_im_g from (8.55) is in complete agreement with the determination of m_i^2 from mass spectroscopy.

8.8 SUMMARY

The essence of Chapter 8 is contained in the expression

$$\langle x', t'|x_0, t_0\rangle = \int_{x_0}^{x'} \mathcal{D}[x(t)]e^{iS[x(t)]/\hbar} \tag{8.56}$$

for the amplitude for a particle at position x_0 at time t_0 to be at position x' at time t'. The right-hand side of (8.56) tells us that the amplitude is proportional to an integral of $e^{iS[x(t)]/\hbar}$ over *all paths* $x(t)$ connecting x_0 to x', subject to the constraint that $x(t_0) = x_0$ and $x(t') = x'$, where

$$S[x(t)] = \int_{t_0}^{t'} dt\, L(x, \dot{x}) \tag{8.57}$$

is the value of the action evaluated for a particular path $x(t)$.

Although evaluating the path integral (8.56) is not especially practical in most problems, the path-integral approach does give us a useful way to think about quantum dynamics. For example, inserting an impenetrable screen with an aperture between a source of particles and a detector, as shown in Figure 8.12*a*, eliminates many of the paths that the particle could follow in moving between the two points, altering the amplitude for the particle to arrive at the detector from what it would have been in the absence of the screen. We call this phenomenon *diffraction.* If a second aperture is opened in the impenetrable screen, as shown in Fig. 8.12*b*, the paths for the particle to reach the detector by traveling through this second slit must be added to the paths to reach the detector by traveling through the first slit, generating an *interference* pattern. In fact, if you have doubts about

[5] Near the surface of the earth $m_gg = Gm_gM/R^2$, where G is the gravitational constant and M is the mass and R the radius of the earth.

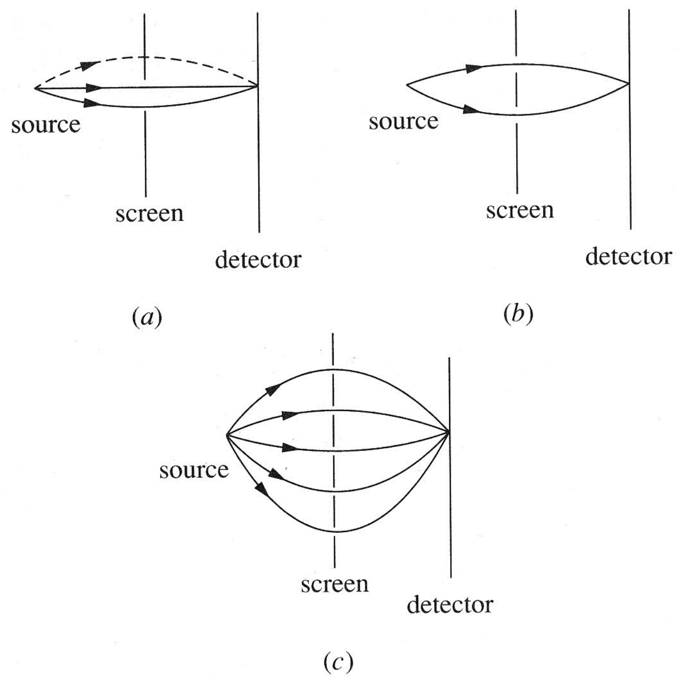


FIGURE 8.12
(*a*) A single-slit diffraction experiment. The path shown with the dashed line is an example of a path that is obstructed by the impenetrable screen and therefore does not contribute to the integral over all paths. (*b*) A double-slit interference experiment. (*c*) A diffraction-grating experiment.

the role played by paths such as the one blocked by the screen in Fig. 8.12*a*, consider opening a periodic array of apertures in the screen to allow the particle following a special set of these paths to reach the detector, as shown in Fig. 8.12*c*. The pattern will clearly differ from that obtained with a single or a double slit.

The path-integral approach also gives insight into the foundations of classical mechanics. Since the factor $e^{iS[x(t)]/\hbar}$ is a complex number of unit modulus, the only thing that differs from one path to another is the value of the phase $S[x(t)]/\hbar$. Figure 8.13 is a schematic diagram of the phase plotted as a function of the path $x(t)$. The particular path where the action is an extremum—$\delta S = 0$—is often called the "path of least action." This path of least action is the unique path $x_{\text{cl}}(t)$ that we expect a particle to follow in classical physics. In quantum mechanics, on the other hand, *all* paths contribute to the path integral (8.56). What makes $x_{\text{cl}}(t)$ special is that since it is the path for which the phase $S[x(t)]/\hbar$ is an extremum, the phase difference between the classical path and its neighbors changes less rapidly than for any other path and its neighbors. When we add up the contribution from all paths, only in the vicinity of the classical path do we find many paths that are in phase with each other and hence can add together coherently. In situations where $S[x(t)]/\hbar \gg 1$, such as for a macroscopic particle, this is a very tight constraint which indeed singles out the classical path and its

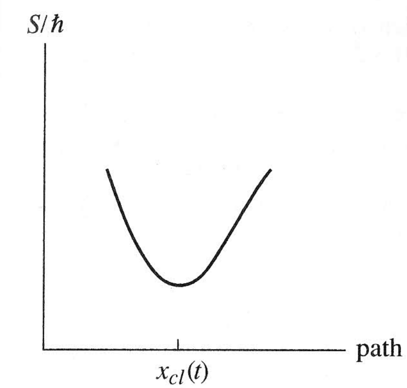


FIGURE 8.13
A schematic diagram of the phase $S[x(t)]/\hbar$ as a function of the path $x(t)$.

very nearby neighbors. However, when $S[x(t)]/\hbar \sim 1$, even paths that deviate significantly from the classical path can still be roughly in phase with it, and the behavior of the particle can no longer be adequately described by classical physics at all.

PROBLEMS

8.1. Use the free-particle propagator (8.8) in (8.3) to determine how the Gaussian position-space wave packet (6.59) evolves in time. Check your result by comparing with (6.76).

8.2. Prove (8.35) by induction.

8.3. Determine, up to an overall multiplicative function of time, the transition amplitude, or propagator, for the harmonic oscillator using path integrals. See Feynman and Hibbs, *Path Integrals and Quantum Mechanics*, Sections 3-5 and 3-6.

8.4. Estimate the size of the action for free neutrons with $\lambda = 1.419$ Å traversing a distance of 10 cm.

8.5. For which of the following does classical mechanics give an adequate description of the motion? Explain.

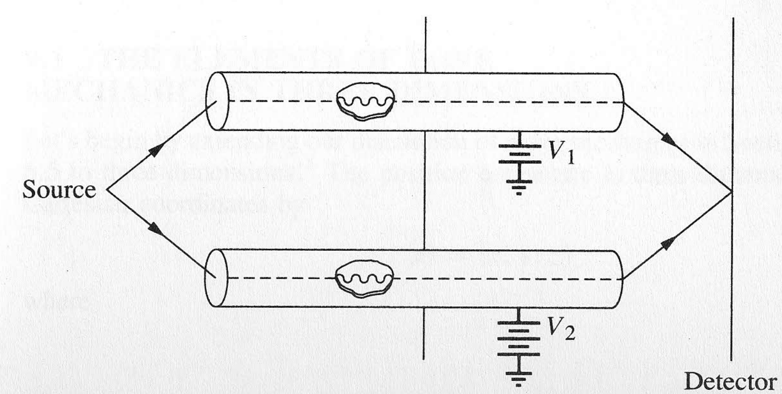


FIGURE 8.14

(*a*) An electron with a speed $v/c = 1/137$, which is typical in the ground state of the hydrogen atom, traversing a distance of 0.5 Å, which is a characteristic size of the atom.

(*b*) An electron with the same speed as in (*a*) traversing a distance of 1 cm.

8.6. A low-intensity beam of charged particles, each with charge q, is split into two parts. Each part then enters a very long metallic tube shown in Fig. 8.14. Suppose that the length of the wave packet for each of the particles is sufficiently smaller than the length of the tube so that for a certain time interval, say from t_0 to t', the wave packet for the particle is definitely within the tubes. During this time interval, a constant potential V_1 is applied to the upper tube and a constant potential V_2 is applied to the lower tube. The rest of the time there is no voltage applied to the tubes. Determine how the interference pattern depends on the voltages V_1 and V_2 and explain physically why this dependence is completely incompatible with classical physics.

CHAPTER 9

TRANSLATIONAL AND ROTATIONAL SYMMETRY IN THE TWO-BODY PROBLEM

After spending Chapters 6, 7, and 8 in one dimension, we now return to the three-dimensional world and consider a system consisting of two bodies that interact through a potential energy that depends only on the magnitude of the distance between them. The Hamiltonian for this system is invariant under translations and rotations of *both* of the bodies, which leads to conservation of total linear momentum and relative orbital angular momentum, respectively. The relationship between an invariance, or a symmetry, in the system and a corresponding conservation law is one of the most fundamental and important in physics.

9.1 THE ELEMENTS OF WAVE MECHANICS IN THREE DIMENSIONS

Let's begin by extending our discussion of wave mechanics in Sections 6.1 through 6.5 to three dimensions.[1] The position eigenstate in three dimensions is given in Cartesian coordinates by

$$|\mathbf{r}\rangle = |x, y, z\rangle \tag{9.1}$$

where

[1] It would be good to review those sections of Chapter 6 before reading Section 9.1.