Notes: Aspects of AdS

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Abstract

Notes of the minicourse "Classical and Quantum Tales on AdS" given in the II MINIWORKSHOP ON FIELD THEORY AND GRATIVATION, UNAM-ICN, México, February 24th-28th, 2020. Still a preliminar version of these notes.

I. INTRODUCTION

Physics on AdS has become particularly relevant in theoretical physics in the context of AdS/CFT correspondence. Before the advent of the correspondence, physics on AdS was studied in the context of gauge supergravities, since the maximally (super)symmetric solution of some of these theories correspond to AdS spacetime, defining the vacuum of the theory. These notes contain certain aspects of physics in AdS. **Homeworks are written with bold.**

II. ADS

A maximally symmetric spacetime, is a spacetime with the maximum number of Killing vectors (globally define), i.e. it has $\frac{D(D+1)}{2}$, independent solutions of

$$L_\xi g_{\mu\nu} = 0 = \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu \xi^\alpha + g_{\nu\alpha}\partial_\mu \xi^\alpha \ . \tag{1}$$

These spacetime are necessarily of constant curvature, i.e. they have the following Riemann tensor

$$R^{\mu\nu}{}_{\lambda\rho} = c\left(\delta^\mu_\lambda \delta^\nu_\rho - \delta^\mu_\rho \delta^\nu_\lambda\right) = c\delta^{\mu\nu}_{\lambda\rho} \ , \tag{2}$$

which implies that the Ricci tensor and the Ricci scalar are

$$R^\mu_\nu = c\,(D-1)\,\delta^\mu_\nu,\ R = cD\,(D-1) \ . \tag{3}$$

The constant c is the curvature of the spacetime, and l with $l^2 = 1/|c|$ is the curvature radius of the spacetime.

Constant curvature spacetimes are necessarily conformally flat, which leads to a vanishing Weyl tensor $C^{\alpha\beta}{}_{\lambda\delta} = 0$ in $D > 3$ and a vanishing Cotton tensor in $C_{\mu\nu} = 0$ in $D = 3$.

Einstein equations with a cosmological constant

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \ , \tag{4}$$

admit a constant curvature solution, provided the curvature of the latter is fixed in terms of the cosmological constant as

$$c = \textbf{Homework} \tag{5}$$

AdS_D is the solution of these equations with $\frac{D(D+1)}{2}$ Killing vectors in the case of negative $c=-\frac{1}{l^2}$.

This spacetime can be obtained as follows. Consider a flat, pseudo-Riemannian space, with the following metric in dimension $D+1$ (notice the absence of x_D just for notational simplicity)

$$ds^2 = -dx_0^2 + dx_1^2 + ... + dx_{D-1}^2 - dx_{D+1}^2 \ . \tag{6}$$

Now, AdS_D is defined as the surface

$$-x_0^2 + x_1^2 + ... + x_{D-1}^2 - x_{D+1}^2 = -l^2 \ , \tag{7}$$

in the spacetime (6). From these equations it is clear the structure of the Killing vectors of the obtained spacetime, since both the ambient metric (6) as well as the constraint defining the surface (7) are invariant under (i,j run from 1 to $D-1$)

$$\begin{aligned}
\text{rotations 1: } & x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i} \text{ there are } \begin{pmatrix} D-1 \\ 2 \end{pmatrix} \\
\text{rotation 2: } & x_0\frac{\partial}{\partial x_{D+1}} - x_{D+1}\frac{\partial}{\partial x_0} \text{ there is 1} \\
\text{boosts 1: } & x_0\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial x_0} \text{ there are } D-1 \\
\text{boost 2: } & x_{D+1}\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial x_{D+1}} \text{ there are } D-1
\end{aligned}$$

There is a total of $\frac{D(D+1)}{2}$ Killing vectors, which can naturally be organized as L_{AB}, with $A,B=0,1,...,D-1,D+1$, and in terms of this, the algebra of the Killing vectors (isometry algebra) reads

$$[L_{AB}, L_{CD}] = \eta_{AD}L_{BC} - \eta_{BD}L_{CA} + \eta_{CA}L_{BD} - \eta_{CB}L_{DA} \ . \tag{8}$$

This is the algebra of the group $SO\left(D-1,2\right)$, with $\eta =$diag$(-1,+1,...,+1,-1)$.

The constraint (7) is one relation between $D+1$ variables, therefore it can be solve introducing D parameters. Substituting a parametrization of the constraint (7) in the metric (6) we find the induced metric on the surface. Different parametrizations will lead to different induced metrics which are related by changes of coordinates. Some of the parametrizations will be global, while other will cover only part of the spacetime. Let us see some particular parametrizations for AdS_3.

Poincare patch parametrization: (t, z, x)

$$x_0 = \frac{z}{2}\left(1 + \frac{(1 + x^2 - t^2)}{z^2}\right),\ x_1 = \frac{t}{z},\ x_2 = \frac{x}{z}\ ,\ x_{D+1} = \frac{z}{2}\left(1 - \frac{(1 - x^2 + t^2)}{z^2}\right)\ . \tag{9}$$

This is a rational parametrization (since the functions are quotients of polynomials), and therefore AdS_D is a rational manifold (not every surface admits a rational parametrization). The induced metric (**homework**) reads

$$ds^2 = \frac{1}{z^2}\left[-dt^2 + dz^2 + dx^2\right]\ . \tag{10}$$

This metric is manifestly conformally flat, and these coordinates do not cover the whole spacetime (one can study analytic extensions as in Schwarzschild black hole).

Global coordinates: (τ, ρ, ϕ)

$$x_0 = \cosh\rho\cos\tau,\ x_1 = \sinh\rho\cos\phi,\ x_2 = \sinh\rho\sin\phi,\ x_{D+1} = \cosh\rho\sin\tau\ . \tag{11}$$

The induced metric reads (**homework**)

$$ds^2 = l^2\left[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\phi^2\right]\ . \tag{12}$$

Notice that in this parametrization τ is periodic, while considering τ periodic on the metric (12) leads to closed timelike curves! Nevertheless, since the metric (12) is well-defined in the range $-\infty < \tau < +\infty$ we are allowed to consider that τ takes values on the whole real line. The obtained spacetime is know as the covering space of AdS, but as usual, hereafter we call this spacetime AdS.

Notice also that the metric near $\rho = 0$ is that of Minkowski3.

AdS in Schwarzschild-like coordinates: (t, r, ϕ)

This is obtained by setting $r = l\sinh\rho$ in the previous parametrization and leads to the following induced metric (**homework**)

$$ds^2 = -\left(\frac{r^2}{l^2} + 1\right)dt^2 + \frac{dr^2}{\left(\frac{r^2}{l^2} + 1\right)} + r^2 d\phi^2\ . \tag{13}$$

All these can be extended to AdS_D to

$$\begin{aligned}
ds^2 &= \frac{1}{z^2}\left[-dt^2 + dz^2 + d\vec{x}^2\right]\ ,\\
ds^2 &= l^2\left[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{D-2}^2\right]\ ,\\
ds^2 &= -\left(\frac{r^2}{l^2} + 1\right)dt^2 + \frac{dr^2}{\left(\frac{r^2}{l^2} + 1\right)} + r^2 d\Omega_{D-2}^2\ ,
\end{aligned}$$

where Ω_{D-2} is the line element of the S^{D-2} (round) sphere. The first parametrization is usually called planar AdS, or AdS in the Poincare patch and does not cover the whole spacetime, while the latter two are referred to as global AdS.

Global structure and Penrose diagrams

The idea behind Penrose diagrams can be understood from the following, simple example. First, the metric of the two-sphere can be written as

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 = \frac{dx^2 + dy^2}{\left(1 + \frac{x^2+y^2}{4}\right)^2} . \tag{14}$$

(**homework: find** $x = x(\theta, \phi)$ and $y = y(\theta, \phi)$ **and extend to sphere of arbitrary radius**). From this we can obtain

$$dx^2 + dy^2 = \left(1 + \frac{x^2(\theta,\phi) + y^2(\theta,\phi)}{4}\right)^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{15}$$

This shows that the metric of the plane is conformal to the metric of the sphere, but notice that the conformal factor blows up as $x^2 + y^2 \to +\infty$. The relations $x = x(\theta, \phi)$ and $y = y(\theta, \phi)$ are the ones obtained by the stereographic projection, where we know that the north pole is associated with all the points of the circle of radius R with $R \to \infty$. Now if we add the north pole we will obtain the sphere, and since the latter is compact, we say that the sphere is the conformal compactification of the plane.

For a spacetime the conformal compactification is constructed in a similar manner, taking care of the fact that there are different types of infinities: the infinite where all the timelike curves end, the infinity where all the null line end and so on.

To study this problem in AdS, lets perform the change of coordinates $r = l\tan x$ in the metric in Schwarzschild-like coordinates, which maps $0 < r < +\infty$ to $0 < x < \frac{\pi}{2}$, leading to the metric

$$ds^2_{AdS_4} = \frac{l^2}{\cos^2 x}\left[-dt^2 + dx^2 + \sin^2 x d\Omega_2^2\right] . \tag{16}$$

This establishes the fact that AdS_4 is conformal to half the Einstein Universe[1]. Consequently AdS has the topology of a solid cylinder, where the axis is located at $x = 0$ and the boundary

[1] EEU is the metric within the brackets in (16) with $0 < x < \pi$, which is globally $R \times S^3$, namely a static cosmology and is solution of Einstein equations with a positive cosmological constant and dust, whose energy density is fixed in terms of the cosmological constant (**homework: prove this**).

at $x \to \frac{\pi}{2}$, while time t runs along the vertical direction, and the surfaces at constant t and constant x are two-spheres S^2. Including $x = \frac{\pi}{2}$ leads to the conformal compactification of the spacetime and now we have a solid cylinder with the boundary included. From this we obtain the Penrose diagram in Figure 1. Notice that the conformal boundary (the mantle of the cylinder) is a timelike surface.

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PENROSE DIAGRAM ADS

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To study a gravitational field (a spacetime) as for a electromagnetic field, it is useful to study probe objects that feel the field. The simplest objects on a spacetime are free test particles which solve the geodesic equation. AdS spacetime has enough symmetries as to lead to a completely solvable geodesic problem. Let us use Schwarzschild-like coordinates to study a particular geodesic. Notice that the coordinate t coincides with the proper time of an observer located at $r = 0$ for all times. Such observer turns out to be geodesic (**homework: prove this**). According to this observer the time it takes for a photon to go from the origin to infinity is

$$\Delta t = \int_0^{+\infty} \frac{dr}{\frac{r^2}{l^2} + 1} = \frac{\pi l}{2} \ . \tag{17}$$

Therefore some extra information must be provided for continuing the history of the photon. One can show (**homework**) that the massive geodesics travelling radially on this spacetime, will return to $r = 0$ in a time πl, regardless the initial radial velocity (as for the harmonic oscillator where the period does not depend of the amplitude). As we increase the initial velocity or decrease the mass, the particle will explore larger distances in AdS, but it will always come back to $r = 0$ in a time $\Delta t = \pi l$. Therefore as a natural continuation of the massive case, we can impose on the massless case that the photon gets reflected when arriving to the boundary and therefore it will return to $r = 0$ in a time πl. This is a reflective boundary condition at infinity for the trajectory of the photon.

Let us now consider a probe field propagating on AdS, and assume for simplicity that the dynamics of the field is that of a massless scalar, i.e. $\Box\Phi = 0$. To solve this problem we must provide initial data on a spacelike surface Σ, which can be seen as the surface $t = t_\Sigma$ for some timelike coordinate t. In flat spacetime, this surface can be choose such that the state of the field Φ at any time later $t = t_\Sigma$ is completely determined by the state of the field (Φ and $\dot{\Phi}$) on the surface Σ. On AdS (see figure) since the conformal boundary is timelike, we must provide further information as a boundary condition for the field, since the boundary is connected with the bulk. The initial value problem on AdS is actually an initial-boundary-value problem.

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CAUCHY PROBLEM ON AdS

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Until now we have considered probe objects on AdS. Now we will make this problem to appear in a perturbative framework of a fully backreackting scenario. Consider for example the Einstein-Klein-Gordon system

$$G_{\mu\nu} - \frac{3}{l^2} g_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \Phi \partial^\alpha \Phi \ , \tag{18}$$

$$\Box \Phi = 0 \ . \tag{19}$$

This theory has an obvious solution which is $g_{\mu\nu} = g_{\mu\nu}^{AdS}$ and $\Phi = 0$. If we perturb this solution by $g_{\mu\nu} = g_{\mu\nu}^{AdS} + \epsilon h_{\mu\nu}$ and $\Phi = 0 + \epsilon\phi$, at the leading order in perturbations we obtain the equation

$$\Box_{AdS} \phi = 0 \ . \tag{20}$$

So we have returned to the previous problem of the probe field on AdS. Let's now move a to a more involved setup, that leads to a similar history in the context of 11D SUGRA.

III. REVIEW OF 11D SUGRA AND SOLUTIONS

11D SUGRA is a field theory with the following field content: g_{MN} (the metric), $A_{MNP} = A_{[MNP]}$ (an Abelian gauge three-form) and a single Majorana spin 3/2 field $\psi_{M,\alpha}$ where the index M transforms as a Lorentz vector while the index α is a Lorentz spinorial Majorana index. The Clifford algebra $\left\{\Gamma^a, \Gamma^b\right\} = 2\eta^{ab}$ in 11 dimensions has a real, Majorana representation with matrices of 32×32 $\left(\Gamma^b\right)^\alpha{}_\beta$. These permit to define a representation of the Lorentz group, which acts on the index α of the spin 3/2 field. There is a match of on-shell degrees of freedom, as required by supergravity, in terms of bosons and fermions (see Appendix I).

The bosonic field equations are

$$R^A_{\ B} = \frac{1}{3} F^{ACDE} F_{BCDE} - \frac{1}{36} \delta^A_B F^2 \ , \tag{21}$$

$$\nabla_A F^{ABCD} = -\frac{1}{576} \epsilon^{BCDM_1 \dots M_8} F_{M_1 \dots M_4} F_{M_5 \dots M_8} \ . \tag{22}$$

Notice that the Einstein equations have been traced, from where one solves the Ricci tensor and replace it back on Einstein equations. Notice also that since the fundamental field is $A_{[3]}$, if we provide an ansatz for $F_{[4]}$, we must ensure that $dF_{[4]} = 0$ which implies the existence of $A_{[3]}$. Via dimensional reduction on $T^7 = S^1 \times ... \times S^1$, using indices $M = \{\mu, i\}$, from the point of view of four-dimensional diffeomorphisms we have

$$g_{MN} \to \{g_{\mu\nu}, g_{\mu i}, g_{ij}\} = \left\{ \text{metric,7 vectors,} \frac{7 \times 8}{2} = 28 \text{ scalars} \right\}$$

$$A_{MNP} \to \{A_{\mu\nu\lambda}, A_{\mu\nu i}, A_{\mu ij}, A_{ijk}\} = \left\{ \text{one 3-form,seven 2-forms,} \frac{7 \times 6}{2} = 21 \text{ vectors, } \begin{pmatrix} 7 \\ 3 \end{pmatrix} = 35 \text{ scalars} \right\}$$

The single 3-form has a field strength that is a 4-form which must be an arbitrary function times the volume form of the four-dimensional spacetime. Maxwell equation then implies that the function must be a constant, and therefore there are no dynamical degrees of freedom coming from $A_{\mu\nu\lambda}$ in four dimensions. On the other hand the seven 2-forms, can be dualized to scalars as follows

$$dA_{[2]} = F_{[3]} = *F_{[1]} = * \left(dA_{[0]} \right) \ , \tag{23}$$

where $A_{[0]}$ are seven scalars. Therefore in total, the dimensional reduction of 11D SUGRA on T^7 leads to a 1 metric, 70 scalars and 28 vectors, as bosonic degrees of freedom.

There is a solution of 11D SUGRA that will be particularly relevant in what follows. The extremal M_2-brane:

$$\begin{aligned} ds^2 &= U^{-2/3} \left[-dt^2 + dx_1^2 + dx_2^2 \right] + U^{1/3} \left(dr^2 + r^2 dS_7^2 \right) \ , \\ F_{[4]} &= dt \wedge dx^1 \wedge dx^2 \wedge dU^{-1} \ , \end{aligned} \tag{24}$$

and $U = U(r) = 1 + \frac{q^6}{r^6}$ is a solution of the equations of 11D SUGRA. Notice that the Chern-Simons term $F_{[4]} \wedge F_{[4]}$ identically vanishes. This metric has isometries $ISO(2,1) \times SO(8)$, which gets enhanced in the solution obtained by taking the near horizon geometry $(r \to 0)$ to $SO(3,2) \times SO(8)$, since the near horizon geometry reads

$$ds^2 = \frac{r^4}{q^4} \left[-dt^2 + dx_1^2 + dx_2^2 \right] + \frac{q^2}{r^2} dr^2 + q^2 dS_7^2 \ , \tag{25}$$

and the submanifold spanned by the coordinates (t, x_1, x_2, r) is AdS_4 of radius q. Of course one has to take also the near horizon geometry in $F_{[4]}$ which turns out to be proportional to the volume form of AdS_4.

This is analogue to what happens in the Majumdar-Papapetrou (1947) solution (that can be though of as a charged $0-$brane) of the Einstein-Maxwell system

$$ds^2 = -U(\vec{x})^{-2} dt^2 + U(\vec{x})^2 d\vec{x}^2, \; A_\mu = U(\vec{x})^{-1} \delta^t_\mu, \tag{26}$$

which is indeed a solution provided $\vec{\nabla}^2 U(\vec{x}) = 0$. Notice that one can superimpose solutions here, which is quite remarkable since the original system is non-linear! If we rewrite $d\vec{x}^2 = dr^2 + r^2 dS_2^2$ and assume $U(r)$, one gets $U(r) = 1 + \frac{q}{r}$. The obtained configuration in this case is actually extremal ($M = Q$) RN solution written in isotropic coordinates (**homework: change the metric to Schwarzschild-like coordinates** $f(\tilde{r}) = \left(1 - \frac{M}{r}\right)^2$). In this isotropic coordinates the horizon is approached as $r \to 0$, and the near horizon geometry of this extremal black hole (as usual with extremal black holes) is $AdS_2 \times S^2$. Even though this metric appears as a near horizon geometry, it defines a configurations that is a global solution of the Einstein-Maxwell system, known as the Bertotti-Robinson spacetime.

Coming back to 11DSUGRA, we can also construct an extremal M_5-brane

$$ds^2 = U^{-1/3} \left[-dt^2 + dx_1^2 + ... + dx_5^2\right] + U^{2/3} \left(dr^2 + r^2 dS_4^2\right) \;, \tag{27}$$

$$F_{[4]} = * \left(dt \wedge dx^1 \wedge ... \wedge dx^5 \wedge dU^{-1}\right) \quad \text{and } U = 1 + \frac{q^3}{r^3} \;. \tag{28}$$

Homework: Isometries of this solutions? What is the geometry on the near horizon ($r \to 0$) **region? What's the contribution of the Chern-Simons term** $F_{[4]} \wedge F_{[4]}$ **to the field equations**?

In the context of the 11DSUGRA the solution (25) is a supersymmetric background. We can define another supersymmetric solution by squashing the S^7 sphere. One can therefore looks for a correlation between the stability properties of the configurations ensured by their supersymmetric nature, and the perturbative stability. The problem of gravitational perturbation is a complex one, and some geometric insight is usually required to simplify, or classify, the perturbations that lead to a tractable set of linearized equations. Using two sets of left-invariants forms of $su(2)$ (see Appendix II) $\{\sigma_1, \sigma_2, \sigma_3\}$ and $\{\Sigma_1, \Sigma_2, \Sigma_3\}$ and their linear combinations $\nu_i = \sigma_i + \Sigma_i$ and $\omega_i = \sigma_i - \Sigma_i$, let's propose the following ansatz that depends on two-functions $u(x^\mu)$ and $v(x^\mu)$ which are scalars from the four-dimensional point of view with coordinates x^μ, namely

$$ds^2 = e^{-7u(x)} g_{\mu\nu} dx^\mu dx^\nu + e^{2u(x)+3v(x)} R_0^2 \left[\frac{d\mu^2}{4} + \frac{1}{16} \sin^2 \mu \omega_i^2 + \frac{e^{-7v(x)}}{16} (\nu_i + \cos \mu \omega_i)^2 \right] . \quad (29)$$

When $v(x) = 0$ the metric within the brackets is that of the round unit S^7 (**homework: compute the Riemann tensor**). Choosing $e^{-7v(x)} = \frac{\sqrt{5}}{5}$ leads to metric within the brackets that describes a space that is topologically a S^7 but it is geometrically deformed (**homework: compute the Riemann and compare it with that of the case** $v = 0$). This defines the squashed S^7 which for this value of $e^{-7v(x)} = \frac{\sqrt{5}}{5}$ leads to an Einstein metric. This is the only value that leads to an Einstein metric on the squashed S^7 **(homework: prove this)**. Of course the round S^7, being a constant curvature space is also an Einstein manifold ($R^i{}_j = c\delta^i_j$ for some c). On top of (29) we must provide an ansatz for $A_{[3]}$, and we propose

$$F_{[4]} = Q e^{-21u(x)} Vol(g_{\mu\nu}) \ , \quad (30)$$

and notice that $dF_{[4]} = 0$ therefore there is and $A_{[3]}$. Here Q is a constant. Assuming $g_{\mu\nu} dx^\mu dx^\nu$ is $AdS_4(L)$ in Schwarzschild-like coordinates we have the following components for $F_{[4]}$

$$F_{MNPQ} = Q e^{-21u(x)} r^2 \sin\theta \times 4! \delta^t_{[M} \delta^r_N \delta^\theta_P \delta^\phi_{Q]} \ . \quad (31)$$

If we first assume $u = u_0$ and $v = v_0$ we found two solutions for the equations of 11DSUGRA.

The round solution: $v = u = 0$, which requires

$$\frac{3}{L^2} = \frac{4Q^2}{3} \text{ and } \frac{1}{R_0^2} = \frac{Q^2}{9} \text{ (round } S^7) \ . \quad (32)$$

The squashed solution: $u = 0$, $v_0 = \frac{1}{7} \ln 5$ (the squashed Einstein S^7 we mentioned before), which requires

$$\frac{3}{L^2} = \frac{4Q^2}{3} \text{ and } \frac{1}{R_0^2} = \frac{25}{81} \frac{Q^2}{5^{4/7}} \text{ (squashed } S^7) \ . \quad (33)$$

Perturbing around each of these solutions ($u = u_0 + \epsilon u_1$ and $v = v_0 + \epsilon v_1$) one obtains at leading order that

$$\text{Round: } \left(\Box_{AdS} - \frac{18}{L^2} \right) u_1 = 0 = \left(\Box_{AdS} - \frac{4}{L^2} \right) v_1 \ , \quad (34)$$

$$\text{Squashed: } \left(\Box_{AdS} - \frac{18}{L^2} \right) u_1 = 0 = \left(\Box_{AdS} + \frac{20}{9L^2} \right) v_1 \ . \quad (35)$$

Notice that the squashing mode v_1, of the squashed solution behaves as a scalar probe on AdS_4 with negative mass squared! Is this in conflict with the supersymmetry of the background?

A similar feature occurs in $N=4$ gauged SUGRA in 4D. The bosonic field content of the theory is a metric, 6 gauge fields for $SO(4)$, one scalar and a pseudo-scalar. The latter can be conveniently packed into a single complex field $z(x) = A(x) + iB(x)$. The truncation to a metric and scalars it is consistent (see Apendix III), leading to a Lagrangian

$$L = R - \frac{\partial_\alpha z \partial^\alpha \bar{z}}{(1 - z\bar{z})^2} - V(z, \bar{z}) \ , \tag{36}$$

with

$$V(z, \bar{z}) = -e^2 \left(3 + \frac{2z\bar{z}}{1 - z\bar{z}}\right) \ , \tag{37}$$

where e is the gravitino charge. The kinetic term is that of a non-linear sigma model on the coset $SU(1,1)/U(1)$, and the potential is invariant only under $U(1)$. Here $z\bar{z} < 1$. Notice that the potential is negative and has a maximum at $z = \bar{z} = 0$. When the scalar vanishes, the field equations reduce to that of AdS with a negative cosmological constant **(Homework: compute the field equations and prove the latter statement)**. Perturbations around this solution lead to a scalar field mode propagating on the AdS background with $L^2 m^2 = -2$.

We need to study the problem of scalar probes on AdS from scratch.

Let us consider a probe scalar field on AdS_4:

$$ds^2 = -\left(\frac{r^2}{L^2} + 1\right) dt^2 + \frac{dr^2}{\frac{r^2}{L^2} + 1} + r^2 + d\Omega_2^2 \ , \tag{38}$$

with the action and equation

$$I[\Phi] = \int d^4x \sqrt{-g} \left[-\frac{1}{2} (\partial\Phi)^2 - \frac{m^2}{2} \Phi^2 - \frac{\xi}{2} \Phi^2 R\right] \ . \tag{39}$$

We have included a conformal coupling with the scalar curvature since the conformally coupled case ($m = 0$ and $\xi = \frac{1}{6}$) has special properties. For these values of the constants $I[\Omega^2(x) g_{\mu\nu}, \Omega^{-1}(x) \Phi] = I[g_{\mu\nu}, \Phi]$. The equation for the scalar field then reads

$$\left(\Box_{AdS} - m^2 - \xi R\right) \Phi = 0 \ . \tag{40}$$

Since the Ricci scalar in AdS is a constant, one can read an effective mass $m_{eff}^2 = m^2 - 12\xi/L^2$. We can introduce a mode superposition

$$\Phi\left(t,r,\theta,\phi\right) = \sum_{l,m} \int d\omega e^{-i\omega t} R\left(r\right)_{l,m,\omega} Y_{l,m}\left(\theta,\phi\right) \ , \tag{41}$$

with $Y_{l,m}\left(\theta,\phi\right)$ spherical harmonics. Here after we set $R\left(r\right)_{l,m,\omega} = R\left(r\right)$. One obtains a second order ODE for the radial function, which has the following, possible, behavior at the origin and near infinity

$$R\left(r\right) = C_1 r^{|l|}\left(1 + O\left(r\right)\right) + C_2 r^{-|l+1|}\left(1 + O\left(r\right)\right) \ , \tag{42}$$

$$R\left(r\right) = D_1 r^{-\Delta_+}\left(1 + O\left(1/r\right)\right) + D_2 r^{-\Delta_-}\left(1 + O\left(1/r\right)\right) \ , \tag{43}$$

with

$$\Delta_\pm = \frac{3}{2}\left(1 \pm \sqrt{1 + \frac{4m_{eff}^2}{9}}\right) \ . \tag{44}$$

Regularity at the origin leads to $C_1 = 0$, while for $m_{eff}^2 > 0$, regularity at infinity requires $D_2 = 0$ since Δ_- is negative in this case, and therefore leads to a non-normalizable solution. The equation of the scaled radial dependence $R\left(r\right) = \tilde{R}\left(r\right)/r^{(D-2)/2}$, leads to the following Schroedinger-like equation

$$-\frac{d\tilde{R}}{dr_*^2} + \frac{L^2 + r^2}{L^4 r^2}\left(\left(L^2 m_{eff}^2 + 2\right) r^2 + l\left(l+1\right) L^2\right) \ \tilde{R} = \omega^2 \tilde{R} \ , \tag{45}$$

where $r = r\left(r_*\right)$ and r_* is the "tortoise coordinate" and therefore

$$dr_* = \frac{dr}{\frac{r^2}{L^2} + 1} \ . \tag{46}$$

The coordinate r_* for AdS runs in the range $[0, \frac{\pi}{2}[$. It is interesting to notice that when the angular momentum of the scalar field vanishes ($l = 0$), the conformally coupled $\xi = \frac{1}{6}$, massless scalar ($m = 0$), leads to the Schroedinger equation (45) for a free particle on the domain $[0, \frac{\pi}{2}[$. The conformal coupling lead to $L^2 m_{eff}^2 = -2$ and therefore the mass $m^2 = -2$ is referred to as the conformal mass.

Connecting the regular behaviours at the origin and nead infinity, leads to the following spectrum

$$L\omega_p = 2p + \Delta_+ + l \text{ with } p = 0, 1, 2, 3... \tag{47}$$

Notice that this spectrum is equispaced, which has important consequences when including the self-interactions or backreaction on the geometry. Each of these modes, labeled by p

have an associated eigenmode e_p. These are usually referred to as AdS oscillons, and as expected from Sturm-Liouville theory it has p nodes. It is interesting that there is a one to one relation beween eigenfunctions with level $N = 2p + l$ and homogeneous polynomials of degree N of the form $P_N\left(X_0^2 + X_5^2, X^i\right)$, on the coordinates of the ambient space of AdS_4 (see 1512.00349).

IV. APENDIX I: COUNTING ON-SHELL DEGREES OF FREEDOM

- Electro: Let's remember first how to compute on-shell degrees of freedom in electromagnetism in arbitrary dimensions (we perform this analysis on flat space). Consider

$$A_M\left(x\right) \sim \int d^D p e^{-ip\cdot x} \tilde{A}_M\left(p\right) \ , \tag{48}$$

and the Fourier transform of the vector field can be written as

$$\tilde{A}_M = a p_M + b \bar{p}_M + c^i \varepsilon^i_M \ , \tag{49}$$

where $p_M = (p_0, \vec{p})$, $\bar{p}_M = (p_0, -\vec{p})$ and ε^i_M are $(D-2)$ spacelike vectors orthogonal to both p and $\bar{p}$. The whole set $\{p, \bar{p}, \varepsilon^i\}$ form a complete basis in momentum space. Two gauge fields that differ by a total derivative $\partial_M \lambda$ are identified (with a λ that decays fast enough at infinity and regular). Clearly in momentum space $\partial_M \lambda \to p_M \tilde{\lambda}$, and therefore the longitudinal component along p_M in (49) can be gauged away, i.e. we can set $a = 0$, and therefore the off-shell number of DOFs is $D-1$ encapsulated in the coefficients b, c^i. Now, we impose Maxwell equations. The field strength reads $F_{MN} = 2\partial_{[M} A_{N]} \to \tilde{F}_{MN} \sim p_{[M}\tilde{A}_{N]}$. Maxwell equation $\partial_M F^{MN} = 0$, therefore leads to

$$\begin{aligned} p^M [p_M\left(b\bar{p}_N + c^i\varepsilon^i_N\right) - p_N\left(b\bar{p}_M + c^i\varepsilon^i_M\right)] = 0 \\ b\left(p^M p_M \bar{p}_N - p^M \bar{p}_M p_N\right) + p^M p_M c^i \varepsilon^i_N = 0 \ , \end{aligned} \tag{50}$$

which due to linear independence leads to $b = 0$ and $p^M p_M = 0$. Therefore we have that the only the $(D-2)$ model along the ε^i_M directions are on-shell degrees of freedom.

- p-form on shell counting: Homework.

Hints: Consider an (Abelian) gauge q-form $A_{[q]} \sim A_{[q]} + d\xi_{[q-1]}$ with $\xi_{[q-1]}$ the gauge parameter. The field strength is $F_{[q+1]} = dA_{[q]}$ and Maxwell equations are $\nabla^\mu F_{\mu\nu_1\ldots\nu_q} = 0$.

Consider Minkowski spacetime and Cartesian coordinates we can Fourier transform $A_{[q]}\left(x\right)$ as

$$A_{[q]}\left(x\right)=A_{M_1\ldots M_q}\left(x\right)\sim\int d^Dpe^{-ip\cdot x}\tilde{A}_{M_1\ldots M_q}\left(p\right)=\int d^Dpe^{-ip\cdot x}\tilde{A}_{[q]}\left(p\right)\ . \tag{51}$$

Given the basis $p_M=(p_0,\vec{p})$, $\bar{p}_M=(p_0,-\vec{p})$ and the $(D-2)$ vectors ε^i_M, one can define the one-forms $p=p_Mdx^M$, $\bar{p}=\bar{p}_Mdx^M$ and $\varepsilon^i=\varepsilon^i_Mdx^M$ and write $\tilde{A}_{[q]}\left(p\right)$ in terms of the basis in the space of $q-$forms, generated by the previous vectors, namely

$$\tilde{A}_{[q]}\left(p\right)=a_{i_1\ldots i_{q-2}}dp\wedge d\bar{p}\wedge\varepsilon^{i_1}\wedge\ldots\wedge\varepsilon^{i_{q-2}}+b_{i_1\ldots i_{q-1}}dp\wedge\varepsilon^{i_1}\wedge\ldots\wedge\varepsilon^{i_{q-1}} \tag{52}$$

$$+\,c_{i_1\ldots i_{q-1}}d\bar{p}\wedge\varepsilon^{i_1}\wedge\ldots\wedge\varepsilon^{i_{q-1}}+d_{i_1\ldots i_q}\varepsilon^{i_1}\wedge\ldots\wedge\varepsilon^{i_q}\ , \tag{53}$$

Impose the gauge redundancy and Maxwell equations to obtain $\begin{pmatrix} D-2 \\ q \end{pmatrix}$ independent components.

- GR symmetric spin 2 field: A symmetric tensor h_{MN} has $\frac{D(D+1)}{2}$ components. Gauge symmetries in this case are given by D diffeomorphism generated by a vector field ξ_M that acts on h_{MN} as $h_{MN}\sim h_{MN}+\partial_M\xi_N+\partial_N\xi_M$. We can therefore substract D components. Consequently the off-shell degrees of freedom are $\frac{D(D-1)}{2}$. Now, Einstein equations contain D constraints, which are now dynamics equations but constraints (the Hamiltonian and the Momentum constraints) on the initial data that will further relate the components of h_{MN}. We end up with $\frac{D(D-3)}{2}$ on-shell degrees of freedom.

- Gravitino: The dynamics of the spin 3/2, Rarita-Schwinger field is

$$\Gamma^{MNP}\partial_N\psi_P=0\ , \tag{54}$$

where ψ_P is a Majorana vector-spinor. Writing explicitly the spinorial index we have $\psi_{P,\alpha}$, and there are $D\times 2^{[D/2]}$ independent components since the the vector index $p=0,...,D-1$ and $\alpha=1,...,2^{[D/2]}$. The equation (54) is invariant under $\psi_P\sim\psi_P+\partial_p\xi$ where ξ is a spinorial parameter, and therefore, if one considers this gauge redundancy one obtains $(D-1)\times 2^{[D/2]}$ off-shell DOFs. ψ_P can be thought of as the gauge connection that allows to promote rigid supersymmetry generated by a global ξ to local supersymmetry generated by a local ξ.

In order to count the on-shell DOFs, let us proceed as follow (see Freedman & VanProyen book). First impose a non-covariant gauge, that completely fixes the gauge freedom

$$\Gamma^i\psi_i=0\ , \tag{55}$$

with $i = 1, ..., D-1$. These are $2^{[D/2]}$ equations and therefore we have, at the moment $(D-1)\times 2^{[D/2]}$. Now consider the following identities for the antisymmetrized products of Dirac matrices (**homework: prove the identities**)

$$\Gamma_M \Gamma^{MNR} = (D-2)\,\Gamma^{NR}, \text{ and } \Gamma^{MNR} = \Gamma^M \Gamma^{NR} - \eta^{MN}\Gamma^R + \eta^{MR}\Gamma^N \ . \tag{56}$$

We can therefore obtain, from equation (54) and the identities the equation (**homework: prove this**)

$$\Gamma^M \left(\partial_M \psi_N - \partial_N \psi_M\right) = 0 \ . \tag{57}$$

Studying the component $N = 0$ and using the gauge-fixing, we get

$$\Gamma^i \partial_i \psi_0 = 0 \ \rightarrow \delta^{ij}\partial_i \partial_j \psi_0 = 0 \ . \tag{58}$$

In order to obtain the last equation we have used the Clifford algebra. The last equation implies that ψ_0 is harmonic in R^3, and since it must be regular and vanishing at infinity we have $\psi_0 = 0$. Remember that $\nabla^2 f(x) = 0 \rightarrow \int d^3x \nabla^2 f(x) = 0 \rightarrow \int d^3x \partial_i f(x)\,\partial^i f(x) + BT = 0$, and if the function f goes to zero sufficiently fast as $|\vec{x}| \rightarrow \infty$ the boundary term will vanish and therefore the norm (positive) of the Euclidean vector $\partial_i f$ must vanish, namely f has to be a constant. The equations $\psi_0 = 0$ allows to substract $2^{\left[\frac{D}{2}\right]}$ extra components.

Inserting this information in the $M = i$ components of (57) one obtains

$$\Gamma^M \partial_M \psi_i = 0 \ , \tag{59}$$

which are $2^{\left[\frac{D}{2}\right]}$ equations.

Therefore since the spinorial equation is of first order, we have

$$\#\text{onshellDOF} = \left[\#\text{components-}\#\text{gaugecondition-}\#\left(\psi_0 = 0\right)\text{-}\#\left(\Gamma^M \partial_M \psi_i = 0\right)\right] \ , \tag{60}$$

$$\frac{(D-3)\times 2^{[D/2]}}{2} = \frac{D \times 2^{[D/2]} - 2^{[D/2]} - 2^{[D/2]} - 2^{[D/2]}}{2} \ . \tag{61}$$

Notice that the gauge freedom plus the dynamics allows to probe $\partial_M\left(\Gamma^M \psi_N\right) = 0$, which projects out the spin $\frac{1}{2}$ combination $\Gamma^M \psi_N$, which is consistent with the fact that Rarita-Schwinger equation describe the dynamics of the irreducible spin $\frac{3}{2}$ field.

V. APENDIX II: ON MAURER-CARTAN FORMS

A general parameter of $SU(2)$ can be written as

$$g = e^{\phi J_3} e^{\theta J_1} e^{\psi J_3} \ , \tag{62}$$

with $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi$ and $0 \leq \psi \leq 4\pi$. This covers once the group manifold. Let's use the following representation for the generators

$$J_1 = \begin{pmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix} \ , \ J_2 = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \ , \ J_3 = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix} \ , \tag{63}$$

which commute as $[J_i, J_j] = \varepsilon_{ijk} J_k$. These generators are anti-hermitian. The inverse of the group element in this parametrization reads

$$g = e^{-\psi J_3} e^{-\theta J_1} e^{-\phi J_3} \ . \tag{64}$$

The differential of the group element $g(\theta, \phi, \psi)$ is defined as

$$dg = \frac{\partial g}{\partial \theta} d\theta + \frac{\partial g}{\partial \phi} d\phi + \frac{\partial g}{\partial \psi} d\psi \ , \tag{65}$$

and the combination $g^{-1} dg$ is a one-form, valued on the algebra, therefore

$$g^{-1} dg = \sigma_i J_i \ , \tag{66}$$

where σ_i are the Maurer-Cartan forms of $SU(2)$. A direct computation (see worksheet) leads to

$$\sigma_1 = \cos\psi d\theta + \sin\theta \sin\psi d\phi \ , \tag{67}$$

$$\sigma_2 = \cos\psi \sin\theta d\phi - \sin\psi d\theta \ ,$$

$$\sigma_3 = d\psi + \cos\theta d\phi \ . \tag{68}$$

You may have seen this structure before, since the kinetic energy of a spinning object with inertia momenta I_1, I_2 and I_3 is

$$K = I_1 \left(\cos\psi \frac{d\theta}{dt} + \sin\theta \sin\psi \frac{d\phi}{dt} \right)^2 + I_2 \left(\cos\psi \sin\theta \frac{d\phi}{dt} - \sin\psi \frac{d\theta}{dt} \right)^2 + I_3 \left(\frac{d\psi}{dt} + \cos\theta \frac{d\phi}{dt} \right)^2 \ , \tag{69}$$

with (θ, ϕ, ψ) the Euler angles (see Landau & Lifshitz V1).

The metric on the squashed three sphere reads

$$ds^2 = \frac{1}{4} \left(I_1 \sigma_1^2 + I_2 \sigma_2^2 + I_3 \sigma_3^2 \right) \ . \tag{70}$$

This is a metric on a space that is topologically a three-sphere, but metrically squashed. The isometry algebras are: $su(2)$ for $I_1 \neq I_2 \neq I_3$, $su(2) \times u(1)$ for $I_1 = I_2 \neq I_3$ and finally $su(2) \times su(2) \sim so(4)$ for $I_1 = I_2 = I_3$. The $\frac{1}{4}$ factor has been included to obtain the unit, three sphere when all the $I_i = 1$.

VI. APENDIX III: BASIC FACTS OF CONSISTENT TRUNCATIONS AND KK REDUCTIONS

It is useful to have in mind the fact that when the action depends linearly in some of the dynamical fields, it is not consistent to set them to zero before computing variations of the action. The reduced action in these cases will lead to equations that are different to that of the full theory after setting the field to zero in the field equations (currents cannot be turn-off in the action). **(Homework: consider the Lagrangian $L = \frac{\dot{q}_1}{2} + \frac{\dot{q}_2}{2} - q_1 q_2^2$ and study the consistency of setting $q_1 = 0$ in the action).**

The dimensional reduction of GR from dimension $D \to D-1$ can be performed by setting the D-dimensional metric as (as in Pope's notes)

$$\hat{g}_{MN} = \begin{pmatrix} e^{2\alpha\phi(x,z)} g_{\mu\nu}(x,z) + e^{2\beta\phi(x,z)} A_\mu(x,z) A_\nu(x,z) & e^{2\beta\phi(x,z)} A_\mu \\ e^{2\beta\phi(x,z)} A_\nu & e^{2\beta\phi(x,z)} \end{pmatrix} . \tag{71}$$

We have split the coordinates as $x^M = \{x^\mu, z\}$. Assuming z is compact, we can introduce a Fourier mode decomposition along that direction. In this case it is consistent to keep only the zero-modes of the expansion, namely assume that the field $g_{\mu\nu}$, A_μ and ϕ, do not depend on z. These fields behave as a metric, a vector field and a scalar field, respectively, with respect to $D-1$-dimensional diffeomorphisms. The dimensional reduction of the Einstein-Hilbert action leads to

$$\frac{1}{16\pi G_D} \int d^D x \sqrt{|\hat{g}|} \hat{R} = \frac{L}{16\pi G_D} \int d^{D-1} x \sqrt{|g|} \left(R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha\phi} F^2 \right) , \tag{72}$$

where we have chosen

$$\alpha^2 = \frac{1}{2(D-1)(D-2)} \text{ and } \beta = -(D-2)\alpha . \tag{73}$$

These latter choice allows to obtain a scalar field that is minimally coupled to gravity, i.e. the $D-1$-dimensional action turns out to be in the Einstein frame, and also the scalar field has a canonical kinetic term. The coupling between the scalar and vector field A_μ is a dilatonic coupling. For generic values of α this theory is known as Einstein-Dilaton-Maxwell **(Homework: Compute the field equations and show that it is inconsistent to set the scalar field to zero, but it is consistent to set the vector field in the action)**