A Brief Introduction to String Theory

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Plan for the mini-course

Motivation

Relativistic bosonic string

Closed string quantization

Quick tour to super string theory

Motivation

Small things, big problems

One of the greatest problems of theoretical physics is the incompatibility of Einstein's General Relativity and the principles of Quantum Mechanics.

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The Standard Model of particle physics, despite its great success, can not be last word, there are a lot of open questions, e.g.:

→ Why so many parameters (more than 20)?

→ Why 26 fields? Why 3 generations?

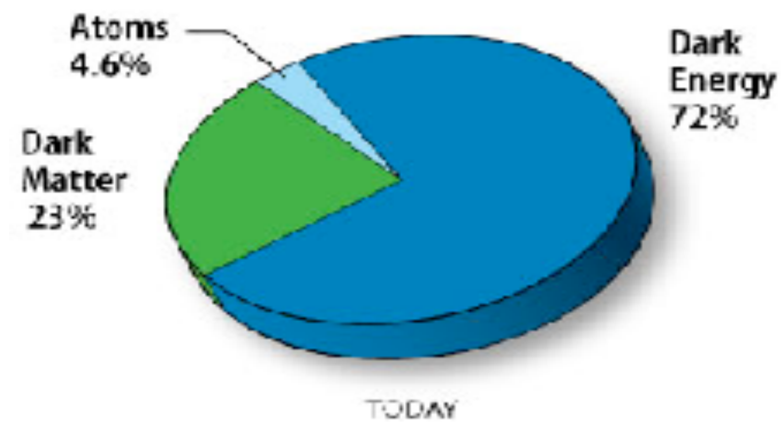
→ Hierarchy problem

→ How do we describe QCD at low energies?

Big things, also big problems

An important question that needs to be answered is what is our universe made of?

- Dark matter (23%)?
- Dark energy (72%)?



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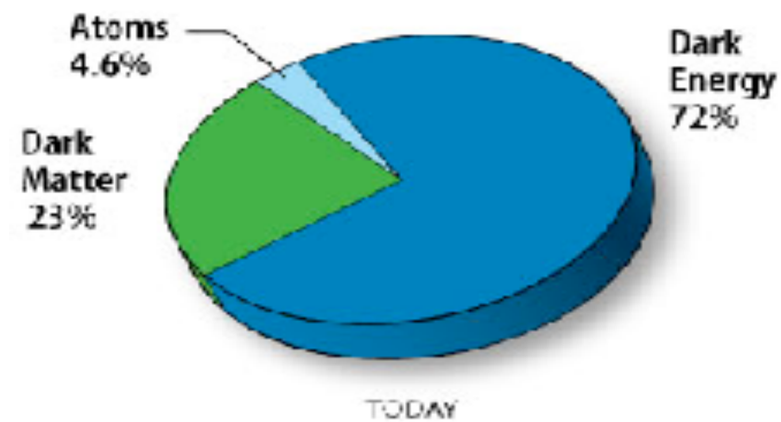


None of these questions seems to have a simple answer

Fortunately, a lot of people with great ideas and very different approaches are trying to solve the puzzles…

One of these roads is STRING THEORY

Why string theory?

(some) Pros:

- String theory is a promising candidate (at least for some people) for the long-sought quantum mechanical theory of gravity.

String theory has the potential to unify the four fundamental forces of nature.

- Interesting new physics (extra dimensions, supersymmetry, more fields, etc)

- A new tool to study certain strongly coupled gauge theories: The AdS/CFT correspondence

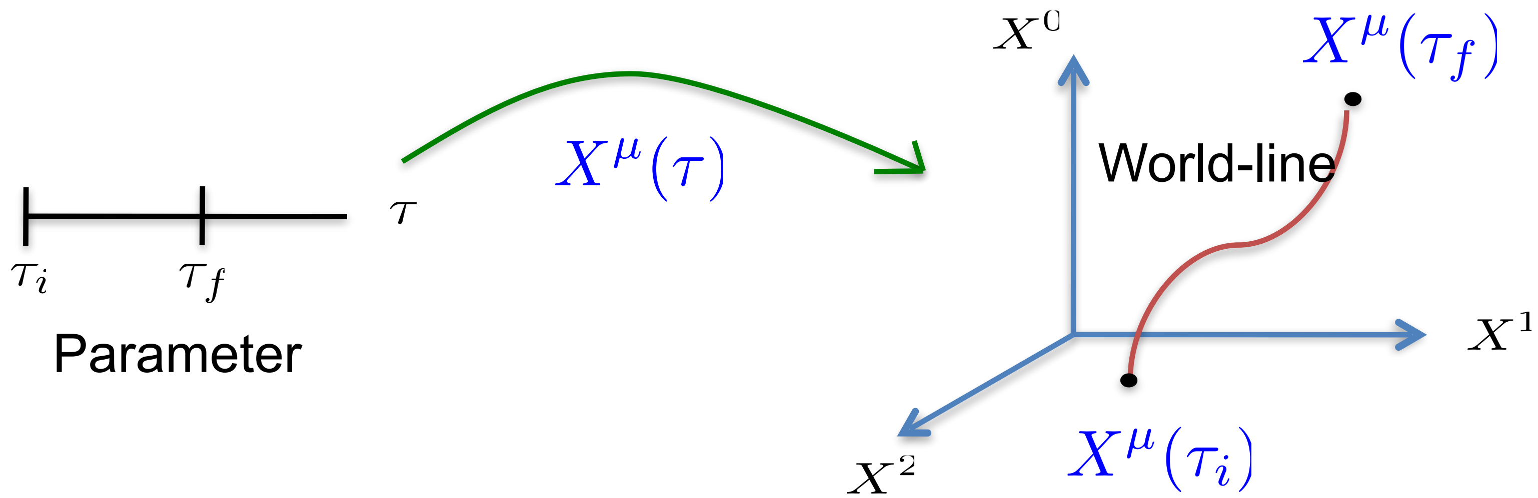
Why string theory?

(some) Cons:

- No direct experimental evidence
- It is far from certain that it describes our world
- String theory has not been able to obtain the Standard Model (similar theories)
- The complete theory still unknown. Lack of a non-perturbative definition
- 10 dimensions?

The relativistic point particle

To preserve manifest Lorentz covariance, we use a parameterized description $X^\mu(\tau)$:



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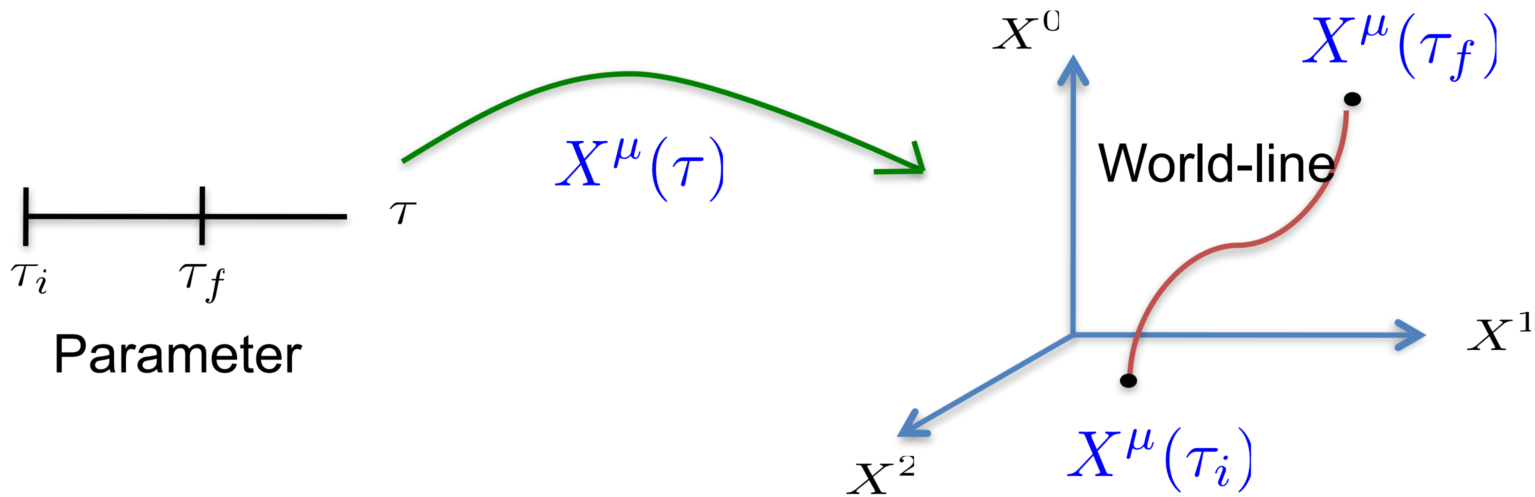

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$$S = -m \times (\text{proper length})$$

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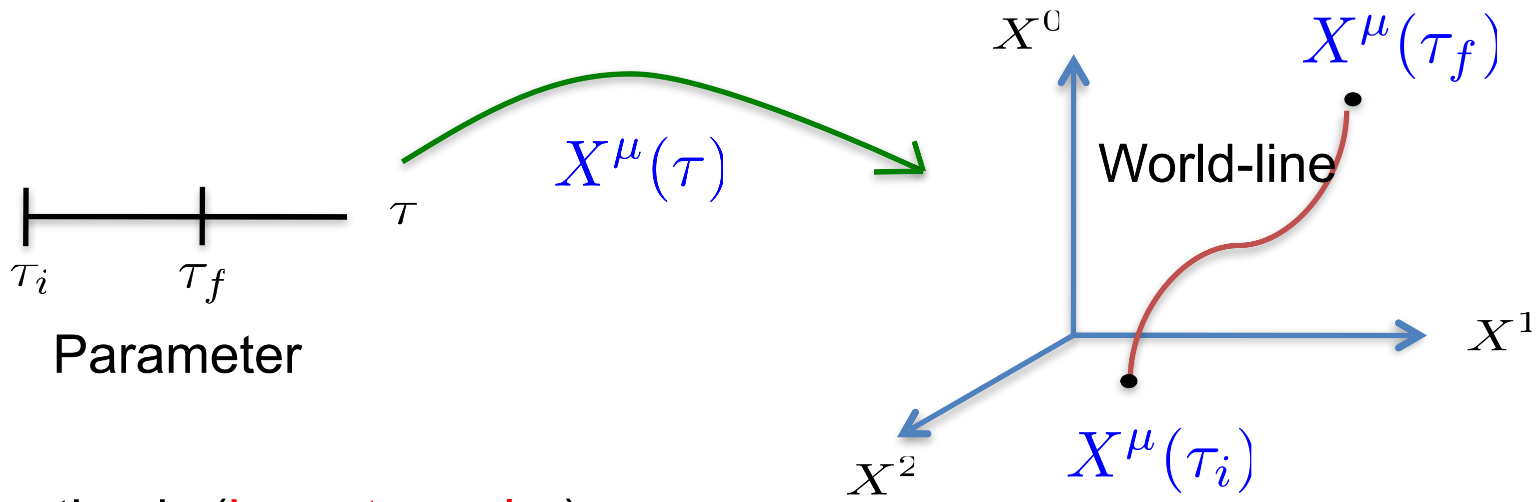

The action is (Lorentz scalar):

$$S[X] = -m \int_{\tau_i}^{\tau_f} d\tau \sqrt{-g_{\tau\tau}} \; = -m \int_{\tau_i}^{\tau_f} d\tau \sqrt{G_{\mu\nu}\partial_\tau X^\mu \partial_\tau X^\nu}$$

where $\quad G_{\mu\nu} = \text{spacetime metric}$

and $\quad g_{\tau\tau} = G_{\mu\nu}\partial_\tau X^\mu \partial_\tau X^\nu = \text{induced metric on worldline}$

The symmetries of this action:

- Spacetime reparametrization invariance (if $G_{\mu\nu} = \eta_{\mu\nu}$ then Poincaré invariance)

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As usual, we define:

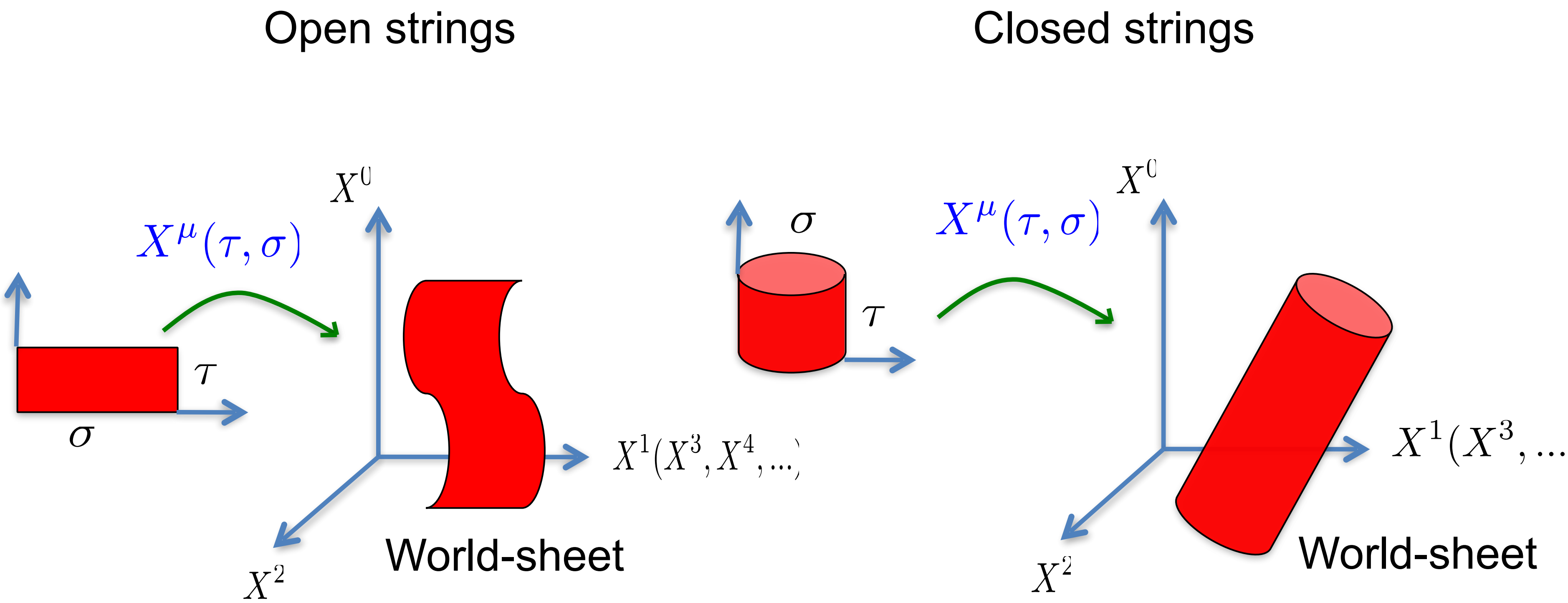
$$P_\mu = \frac{\partial L}{\partial(\partial_\tau X^\mu)} = \frac{-m\dot{X}^\mu}{\sqrt{-\dot{X}^2}}$$

and P_μ satisfies the condition: $P_\mu P^\mu + m^2 = 0$ (first class const.)

$\longrightarrow$ D-1 degrees of freedom.

Can we generalise this to a 1-dimensional object??

The relativistic bosonic string



The world-sheet is described by the embedding functions: $X^\mu(\tau,\sigma)$

And in complete analogy with the relativistic point particle:

$$S[X] = -T \times (\text{proper area}) \qquad (T \equiv \text{tension})$$

$$\boxed{S_{NG}[X] = -T \int d\tau d\sigma \sqrt{-\det g_{ab}}}$$

where $g_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \equiv \text{induced metric on the worldsheet}$

and

$T \equiv \dfrac{\text{energy}}{\text{length}} \equiv \dfrac{1}{2\pi l_s^2}$ where l_s is the fundamental string length

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What are the symmetries of this action?

The symmetries of the Nambu-Goto action:

- Spacetime reparametrization invariance (if $G_{\mu\nu} = \eta_{\mu\nu}$ then Poincaré invariance)

- Worldsheet reparametrization invariance $(\tau, \sigma) \rightarrow (\tau'(\tau, \sigma), \sigma'(\tau, \sigma))$

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For the rest of the talk we will consider $G_{\mu\nu} = \eta_{\mu\nu}$, so the NG action is given by

$$S_{NG} = -T \int d\tau d\sigma \sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 \left(X'\right)^2}$$

The Nambu-Goto action is non-polynomial, so it is convenient to work with what is known as the Polyakov action

$$S_p[X, h_{ab}] = -T \int d\tau d\sigma \sqrt{-h} h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\mu \qquad (h \equiv \det h_{ab})$$

where $h_{ab} \equiv \text{intrinsic metric on the worldsheet}$, and the other elements are the same as before.

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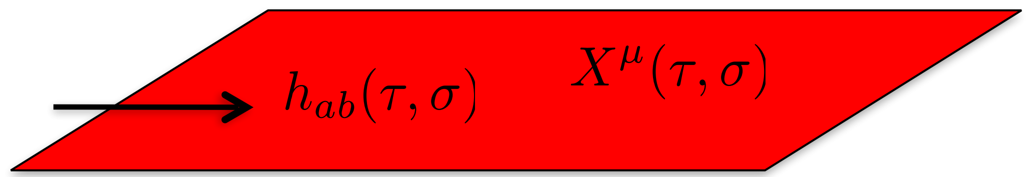
Auxiliary variable on the world-sheet
(Lagrange multiplier)



Open string world-sheet

- The intrinsic metric is a dynamical field on the string world-sheet.
- At the classical level the Polyakov action is equivalent to the NG action.

The symmetries of the Polyakov action:

- Spacetime reparametrization invariance (if $G_{\mu\nu} = \eta_{\mu\nu}$ then Poincaré invariance)

- Worldsheet reparametrization invariance $(\tau, \sigma) \to (\tau'(\tau, \sigma), \sigma'(\tau, \sigma))$

- Weyl invariance $h'_{ab}(\tau, \sigma) = \Omega(\tau, \sigma) h_{ab}(\tau, \sigma)$

Before going further, let's pause and say a few words about symmetries and anomalies.

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→ Anomalies in gauge symmetries are fatal!

The Weyl symmetry suffers an anomaly and this will play a central role in string theory.

In particular, “fixing this problem” implies that the theory lives in 26 space-time dimensions! (more on this later).

Now, let's work the equations of motion (eom).

- Notice that now we have to vary with respect to $X^\mu(\tau,\sigma)$ AND h_{ab}.

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The equation of motion for h_{ab}:

$$\frac{\delta S_P}{\delta h^{ab}} = 0 \quad \longrightarrow \quad \partial_a X^\mu \partial_b X^\nu - \frac{1}{2}h_{ab}h^{cd}\partial_c X^\mu \partial_d X^\nu = 0$$

This last equation can be rewritten as

$$h_{ab}(\sigma,\tau) = \lambda(\sigma,\tau)g_{ab}(\sigma,\tau)$$

i.e. the intrinsic metric is proportional to the induced metric!

Remember that $T_{ab} \equiv \frac{4\pi}{\sqrt{-g}} \frac{\delta S}{\delta h^{ab}}$, meaning that the variation of the action with respect to the intrinsic metric is the energy-momentum tensor on the string world-sheet.

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And what about the kinetic term for h_{ab}?

→ Turns out that gravity in 1+1 dimensions is non-dynamical (due to diffeos. invariance and Bianchi identities), therefore the LHS of Einstein's equations is identically zero.

→ Let's see what happens if we choose $h_{ab} = \eta_{ab}$

Rewriting the Polyakov action with $G_{\mu\nu} = \eta_{\mu\nu}$ and $h_{ab} = \eta_{ab}$:

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$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu(\tau, \sigma) = 0$$ (wave equation!)

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$$\begin{cases} T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0 \\ T_{01} = \dot{X} \cdot X' = 0 \end{cases}$$

To sum up, the dynamics of the string is govern by the wave equation subject to two constraints ($T_{ab} = 0$)

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\left\{
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Degrees of freedom? $\longrightarrow$ $D\,\text{variables} - 2\,\text{constraints} = (D-2)\,\text{dof}$

What is the physical meaning of these constraints?

1. $\dot{X} \cdot X' = 0$ the motion of the string (its velocity) is perpendicular to the string itself.

2. $\dot{X}^2 + X'^2 = 0$, do it as an exercise!

Boundary conditions

Closed string: $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi)$ $\quad 0 \leqslant \sigma \leqslant 2\pi$ (Periodic)

Open string: $0 \leqslant \sigma \leqslant \pi$

1- Covariant under Poincaré:

$$\partial_\sigma X^\mu(\tau,0) = \partial_\sigma X^\mu(\tau,\pi) = 0 \quad \forall \;\; \tau$$ (Neumann: free endpoints)

2- Non covariant under Poincaré

$$\partial_\tau X^i(\tau,0) = \partial_\tau X^i(\tau,\pi) = 0 \quad \forall \;\; \tau \;\; i = p+1, \ldots, D-1$$

$$\longrightarrow \quad X^i(\tau,0) = X^i(\tau,\pi) = c^i$$ (Dirichlet: fixed endpoints)

This last case has very important implications

(Curso de Oscar)

Closed string quantization

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- → We will construct the Fock space.
- → Following this procedure, we will obtain states with negative norm (call ghost) and we will have to use the constraints in order to get the physical spectrum (Gupta-Bleuler method).

Closed string quantization

Something to keep in the back of your head:

When we quantize a theory (or a system), it is not always the case that the quantum equations of motion are the same as the classical ones. In principle, you should first obtain the time evolution operator (i.e. the Hamiltonian) and evolve the corresponding fields using the commutator. E.g.

$$[H, \phi(x)] = -i\dot{\phi}(x)$$

Luckily, in our case, the classical eom coincide with their quantum version (we will not prove it) so we don't have to worry about that.

Closed string quantization

We want to solve the e.o.m. $\partial^2 X^\mu(\tau, \sigma) = 0$

subject to periodic boundary conditions $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi), \forall\, \tau,\, \sigma$

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It is not difficult to show that the solution is given by

$$X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu \tau + i\sqrt{\frac{l_s^2}{2}} \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^\mu e^{-in(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau-\sigma)}\right)$$

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$$X^\mu(\tau,\sigma) = \underbrace{x^\mu}_{\text{Position of Center of mass}} + l_s^2 \underbrace{p^\mu}_{\text{Momentum of center of mass}} \tau + i\sqrt{\frac{l_s^2}{2}} \sum_{n\neq 0} \frac{1}{n} \underbrace{\left(\alpha_n^\mu e^{-in(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau-\sigma)}\right)}_{\text{String oscillations}}$$

Right-moving mode

Left-moving mode

and $p^\mu = \sqrt{\frac{2}{l_s^2}}\,\alpha_0^\mu$, i.e. the zero mode is proportional to the spacetime momentum of the string.

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Discrete momentum: $p = n$
(circle)

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And we postulate: $$\left[\hat{X}^\mu(\tau,\sigma),\hat{\Pi}^\nu(\tau,\sigma)\right] = i\eta^{\mu\nu}\delta(\sigma-\sigma')$$

(Completely analogous to what you've done in Alberto's lectures)

Plugging our solution into the commutation relation and after doing some algebra we get

$$\begin{cases} [\hat{x}^\mu,\hat{p}^\nu] = i\eta_{\mu\nu} \\ [\hat{\alpha}^\mu_m,\hat{\alpha}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0} \\ [\hat{\tilde{\alpha}}^\mu_m,\hat{\tilde{\alpha}}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0} \end{cases}$$

identical to:

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And now we can construct the Fock space...

As usual, let's define the vacuum state $| \, 0, 0; k \rangle$ such that:

$$\alpha_n^\mu \, | \, 0, 0; k \rangle = 0 = \tilde{\alpha}_n^\mu \, | \, 0, 0; k \rangle \qquad \forall \;\; n > 0$$

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For example, some of the states are:

$$\underbrace{|\,0,0;k\rangle}_{\text{Vacuum: no oscillators}} \qquad \underbrace{(\alpha_n^\mu)^\dagger\,|\,0,0;k\rangle}_{\text{One left-moving oscillator}} \qquad \underbrace{(\alpha_n^\mu)^\dagger(\alpha_m^\nu)^\dagger\,|\,0,0;k\rangle}_{\text{Two left-moving oscillators}}$$

(We've made a small change of notation: $\alpha^\mu{}_{-n} = (\alpha_n^\mu)^\dagger$ and $\tilde{\alpha}^\mu{}_{-n} = (\tilde{\alpha}_n^\mu)^\dagger$)

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We have not used the constraints yet, so, as we mentioned earlier we should expect to find states with NEGATIVE NORM.

An example:

$$(\alpha_n^0)^\dagger \mid 0,0;k\rangle \qquad \forall \;\; n>0$$

$$\mid (\alpha_n^0)^\dagger \mid 0,0;k\rangle \mid^2 = \langle 0,0;k \mid \underbrace{\alpha_n^0(\alpha_n^0)^\dagger}_{[\alpha_n^0,(\alpha_n^0)^\dagger] = n\eta^{00} = -n} \mid 0,0;k\rangle = -n(2\pi)^D\delta^D(0)$$

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In order to get rid of this problem we need to use the constraints.

This means:

$$\underbrace{T_{ab}|\psi\rangle = 0}_{\text{This is just the quantum version of the constraints}}$$

(analogue to the Gupta-Bleuler method for the Maxwell field)

First, recall: $T_{ab} = g_{ab} - \frac{1}{2} h_{ab} h^{cd} g_{cd} = 0$ and $T^a_a = 0$ (Weyl invariance)

Let's define: $\begin{cases} \sigma^+ \equiv \tau + \sigma \\ \sigma^- \equiv \tau - \sigma \end{cases}$ and rewrite T_{ab} :

$$T_{++} = \frac{1}{2}(T_{00} + T_{01}) = \partial_+ X^\mu \partial_+ X_\mu$$

$$T_{--} = \frac{1}{2}(T_{00} - T_{01}) = \partial_- X^\mu \partial_- X_\mu$$

$$\xrightarrow{\partial_a T_{ab} = 0} \begin{cases} T_{++} \equiv T_{++}(\sigma^+) \\ T_{--} \equiv T_{--}(\sigma^-) \end{cases}$$

First, recall: $T_{ab} = g_{ab} - \frac{1}{2} h_{ab} h^{cd} g_{cd} = 0$ and $T^a_a = 0$ (Weyl invariance)

Let's define: $\left\{ \begin{array}{l} \sigma^+ \equiv \tau + \sigma \\ \sigma^- \equiv \tau - \sigma \end{array} \right.$ and rewrite T_{ab} :

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$$\xrightarrow{\partial_a T_{ab} = 0} \left\{ \begin{array}{l} T_{++} \equiv T_{++}(\sigma^+) \\ T_{--} \equiv T_{--}(\sigma^-) \end{array} \right.$$

and now, we can expand $T_{++}(\sigma^+)$ and $T_{--}(\sigma^-)$ in Fourier modes, i.e.,

$$T_{++}(\sigma^+) = \sum_{n=-\infty}^{n=\infty} L_n e^{-in\sigma^+}$$

$$T_{--}(\sigma^-) = \sum_{n=-\infty}^{n=\infty} \tilde{L}_n e^{-in\sigma^-}$$

It is possible to write the Fourier coefficients in terms of the left and and right-moving modes (working session):

$$L_m = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n \quad \text{and} \quad \tilde{L}_m = \frac{1}{2}\sum_{-\infty}^{\infty} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n$$

These are called the Virasoro operators and you will here a lot about them if you study string theory or CFT's.

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At this point, we should remember (or learn) that in quantum mechanics it is important to consider "ordering" when writing products of operators (due to the commutation relation). An example of this is the NORMAL ORDER, which we define in the following way:

$$:\alpha_m^\mu \alpha_n^\nu: \;:= \begin{cases} \alpha_m^\mu \alpha_n^\nu, & \text{if}\, m \leq n \\ \alpha_n^\nu \alpha_m^\mu, & \text{if}\, n < m \end{cases}$$

(i.e. we put the creation operators to the left)

Then, we redefine

$$L_m = \frac{1}{2}\sum_{n\epsilon\mathbb{Z}} :\alpha_{m-n} \cdot \alpha_n: \quad \text{and} \quad \tilde{L}_m = \frac{1}{2}\sum_{n\epsilon\mathbb{Z}} :\tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n:$$

Given the definition of $L_m\,(\tilde{L}_m)$, notice that $L_0\,(\tilde{L}_0)$ is the only operator with an ambiguous ordering.

We write (and the same is true for $\tilde{L}_0$)

$$L_0 = \frac{1}{2}\alpha_0 \cdot \alpha_0 + \underbrace{\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n}_{N \,\equiv\, \text{number operator}}$$

Explicitly,

$$L_0 = \frac{1}{2}\alpha_0 \cdot \alpha_0 + \frac{1}{2}\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \frac{1}{2}\sum_{n=1}^{\infty} \alpha_n \cdot \alpha_{-n}$$

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The first and second term have the correct order, however the third one does not. Using the commutation relations:

$$\frac{1}{2}\sum_{n=1}^{\infty}\alpha_n \cdot \alpha_{-n} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\alpha_{-n} \cdot \alpha_n + [\alpha_n, \alpha_{-n}]\right)$$

Then,

$$L_0 = \frac{1}{2}\alpha_0 \cdot \alpha_0 + \sum_{n=1}^{\infty}\alpha_{-n} \cdot \alpha_n + \frac{(D-2)}{2}\sum_{n=1}^{\infty} n$$

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This is the space-time dimension

Explicitly,

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Have fun, and show that this converges to $-\frac{1}{12}$!!

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So finally,

$$\boxed{L_0 = \frac{1}{2}\alpha_0 \cdot \alpha_0 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n - \frac{(D-2)}{24}}$$

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When quantizing the classical expression we will need to introduce a normal ordering constant "a", then

$$\boxed{L_0 \rightarrow L_0 - a}$$

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Turns out that a ghost free spectrum is only possible for certain values of the constant a and the space-time dimension D .

If time allows, we will show that the critical values are:

$$\boxed{a = -1 \quad \text{and} \quad D = 26}$$

Before we use the constraints to find the spectrum of the closed string, let's say a few more things about the Virasoro operators.

They satisfy the quantum Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

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→ c is call the central charge and it can be shown that it is given by the number of space-time dimensions (see GSW pp. 81).

→ The fact that $c \neq 0$ indicates a quantum anomaly (at the classical level $c \equiv 0$).

This is precisely the anomaly that we mentioned before and when fixing this problem we find the space-time dimension of string theory.

Finally the quantum constraints $T_{ab}|\psi\rangle = 0$, can be written in terms of the Visaroro operators in the following way:

$$\begin{cases} (L_0 - 1)|\psi\rangle = 0 = (\tilde{L}_0 - 1)|\psi\rangle \\ L_{m>0}|\psi\rangle = 0 = \tilde{L}_{m>0}|\psi\rangle \\ (N - \tilde{N})|\psi\rangle = 0 \quad \text{(level matching condition)} \end{cases}$$

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Remember that we can write,

$$L_0 = \frac{l_s^2}{4}p^2 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n = \frac{l_s^2}{4}p^2 + N \quad \text{and} \quad \tilde{L}_0 = \frac{l_s^2}{4}p^2 + \sum_{n>0} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = \frac{l_s^2}{4}p^2 + \tilde{N}$$

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and using the mass shell condition $M^2 = -p^2$ (and the above constraints) we find

$$\boxed{M^2 = \frac{2}{l_s^2}(N + \tilde{N} - 2)}$$

The closed string spectrum

Starting from $M^2 = \frac{2}{l_s^2}(N + \tilde{N} - 2)$

1. $N = \tilde{N} = 0$ $\longrightarrow$ $M^2 = -\frac{4}{l_s^2}$

State: $|0; k\rangle$ (No oscillators acting on the ground state).

Scalar field $T(x)$ with negative mass squared called TACHYON

(sign of instability because $\frac{d^2V(T)}{dT^2} < 0$)

The closed string spectrum

Starting from $M^2 = \frac{2}{l_s^2}(N + \tilde{N} - 2)$

2. $N = 1 = \tilde{N} \longrightarrow M^2 = 0$

States: $\epsilon_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0,0;k\rangle$ with $k^2 = 0$

Now, we use the second constraint,

$$L_1(\epsilon_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0,0;k\rangle) \propto (\alpha_0 \cdot \alpha_1 + \cdots +)(\epsilon_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0,0;k\rangle) = 0$$

(the same with $\tilde{L}_1$)

and the result is

$$\boxed{\text{physical state} \longleftrightarrow k^{\mu}\epsilon_{\mu\nu} = 0 = k^{\nu}\epsilon_{\mu\nu}}$$

Finally, the most general physical state is given by

$$\boxed{\epsilon_{ij}\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0,0;k\rangle \quad \text{with} \quad i,j=2,\ldots,D-1}$$

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The matrix can be split into 3 different cases:

1. Trace: $\epsilon_{ij} \propto \delta_{ij} \longrightarrow$ spinless particle;

   1 state, scalar field called the dilaton $\phi(x)$

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2. Symmetric (traceless) part: $\epsilon_{(ij)}$ $\longrightarrow$ spin 2 particle;

   $\frac{(D-2)(D-1)}{2} - 1$ states; the graviton $h_{\mu\nu}(x)$ $(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu})$

This is why (some) people say that string theory is a candidate for quantum gravity!

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3. Antisymmetric part: $\epsilon_{[ij]}$

   $\frac{(D-2)(D-3)}{2}$ states; called Kalb-Ramond field $B_{\mu\nu}(x)$

The first massive state,

$$N = \tilde{N} \geq 2 \longrightarrow M^2 \geq \frac{4}{l_s^2} \quad \text{Very heavy!!!}$$

… With this, we have finished with the closed string.

What about the open string quantization?

Open string quantization

Again, want to solve $\partial^2 X^\mu(\tau,\sigma) = 0$, now with Neumann b.c.

i.e., $\partial_\sigma X^\mu(\tau,0) = \partial_\sigma X^\mu(\tau,\pi) = 0 \quad \forall \quad \tau$

As a consequence of b.c. : $\alpha_n^\mu = \tilde{\alpha}_n^\mu \quad \forall \quad n$ (stationary wave)

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As a consequence of b.c. : $\alpha_n^\mu = \tilde{\alpha}_n^\mu \quad \forall \quad n$ (stationary wave)

The solution to the e.o.m. :

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos n\sigma$$

The quantization process is exactly the same as for the closed string

(But just with one set of oscillators)

Let's jump directly to the spectrum…

The constraints translate into:

$$\begin{cases} (L_0 - 1)|\psi\rangle = 0 \\ L_{m>0}|\psi\rangle = 0 \end{cases}$$

Using the first condition: $$\boxed{M^2 = \frac{N-1}{l_s^2}}$$

$$N = 0 \longrightarrow M^2 = -\frac{1}{l_s^2}$$ State: $|0;k\rangle$

Open string tachyon field $t(x)$

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$N = 1 \longrightarrow M^2 = 0$ States: $\epsilon_\mu \alpha^\mu_{-1}|0;k\rangle$

If $\epsilon_\mu k^\mu = 0 \longrightarrow$ physical state: $\epsilon_i \alpha^i_{-1}|0;k\rangle \;\; \text{with} \;\; i = 2, \ldots, D-1$

$(D-2)$ states of a spin 1 particle $\longleftrightarrow$ Massless vector field $A_\mu(x)$

To sum up:

<u>Closed string spectrum</u>:

- Tachyon field $T(x)$
- Dilaton $\phi(x)$
- Graviton $h_{\mu\nu}(x)$
- Kalb-Ramond field $B_{\mu\nu}(x)$
- Infinite tower of massive fields (very heavy!)

<u>Open string spectrum</u>:

- Tachyon field $t(x)$
- Maxwell field $A_\mu(x)$
- Scalar fields (Dirichlet b.c.) $\phi^I(x)$
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What about interactions?

(Comments on the blackboard)

Before we continue, let me say that by studying how all these fields interact it is possible to construct and effective action.
In particular, for the massless modes of the closed superstring:

$$S = \frac{1}{(2\pi)^7 g_s^2 l_s^8} \int d^{10}x \sqrt{g}[R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}e^{-\phi}H_3^2 - \frac{1}{2}e^{2\phi}F_1^2 - \frac{1}{2}e^{\phi}F_3^2 - \frac{1}{4}F_5^2] - \frac{1}{(2\pi)^7 g_s^2 l_s^8} \int C_4 \wedge H_3 \wedge F_3,$$

Note: don’t worry about the details…

This is known as the supergravity action (and corresponds to the low energy $(E \ll 1/l_s)$ limit of Type IIB string theory).
There is also an affective action for the massless modes of the open string (Curso de Oscar).

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Notice, that besides having tachyons, the are NO FERMIONS in the bosonic string!

For that, we need SUPERSTRING THEORY!

A quick tour to Superstring Theory

We have learnt that the bosonic string theory has at least three important problems:

1. Tachyons (i.e. scalar fields with negative mass).
2. No fermions in the spectrum.
3. The space-time dimension is 26.

A quick tour to Superstring Theory

We have learnt that the bosonic string theory has at least three important problems:

1. Tachyons (i.e. scalar fields with negative mass).

2. No fermions in the spectrum.

3. The space-time dimension is 26.

The situation gets better when we incorporate a new symmetry to the theory, meaning SUPERSYMMETRY.

Boson $\longleftrightarrow$ Fermions

(Curso Elena)

There are 3 equivalent formalism to describe superstring theory.

A quick tour to Superstring Theory

1. Green-Schwarz

- → We incorporate new variables $\theta^{A}_{\alpha}(\sigma)$ $\begin{cases} \text{space-time spinor} \\ \text{worldsheet scalar} \end{cases}$
- → Manifestly space-time susy
- → It can only be quantize in the light cone gauge.

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2. Ramond-Neveu-Schwarz

- We incorporate new variables $\psi^\mu_A(\sigma)$ $\left\{\begin{array}{l}\text{space-time vector}\\ \text{worldsheet spinor}\end{array}\right.$
- Not manifestly space-time susy
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3. Pure spinor formalism (Berkovitz)

→ Still under construction

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Ramond-Neveu-Schwarz formalism

Superstring theory is obtained by adding to the bosonic string, whose action in flat gauge, we have seen is

$$S_B = -T \int d^2\sigma \partial_a X^\mu \partial^a X_\mu$$

with X^μ a worldsheet scalar, a sector describing 2-dimensional worldsheet spinors.

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→ Remember (or learn) that a spinor is by definition a representation of the Clifford algebra. Applied to the 2 dimensional worldsheet with flat metric, the Clifford algebra is generated by two dimensional $\gamma - \mathrm{matrices}$ with anti-commutation relations

$$\{\gamma^a, \gamma^b\}_{AB} = 2\eta^{ab}\mathbb{I}_{AB} \quad \text{where} \quad \gamma^a = \gamma^a_{AB}$$

Here $A\,, B$ are spinor indices on the worldsheet and $a\,, b$ are vector indices ($a\,, b = 0\,, 1$).

Explicitly, in 2-dimensions the gamma matrices are given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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→ A spinor ψ^A transform under Lorentz transformations as

$$\psi_A \to S_{AB}\psi_B, \quad S_{AB} = \left[\exp(i\omega_{ab}\frac{i}{4}[\gamma^A, \gamma^B])\right]$$

with ω_{ab} a Lorentz infinitesimal transformation.

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→ In view of this, ψ^A can be taken to be real.

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \qquad \psi^* = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}^* = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

This REALITY condition is called Majorana condition and the corresponding spinor is called Majorana spinor.

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The labelling $\psi_\pm$ refers to the chirality, i.e. the eigenvalues under $\gamma = \gamma^0\gamma^1$

We can now write the RNS action in flat gauge as the action obtained by adding the canonical term for free bosons and Majorana fermions on the worldsheet

$$\boxed{S = -T \int d^2\sigma \left(\partial_a X^\mu \partial^a X_\mu + i \bar{\psi}^\mu_A \gamma^a_{AB} \partial_a \psi_{\mu B} \right)}$$

where $\psi^\mu_A = \begin{pmatrix} \psi^\mu_+ \\ \psi^\mu_- \end{pmatrix}$ with $\psi^\mu_\pm$ representing Grassman valued space-time vectors, and $\bar{\psi} = \psi^\dagger \gamma^0 = (-\psi_-, \psi_+)$.

We can now write the RNS action in flat gauge as the action obtained by adding the canonical term for free bosons and Majorana fermions on the worldsheet

$$\boxed{S = -T\int d^2\sigma\big(\partial_a X^\mu \partial^a X_\mu + i\bar{\psi}^\mu_A \gamma^a_{AB}\partial_a \psi_{\mu B}\big)}$$

where $\psi^\mu_A = \left(\begin{array}{c}\psi^\mu_+ \\ \psi^\mu_-\end{array}\right)$ with $\psi^\mu_\pm$ representing Grassman valued space-time vectors, and $\bar{\psi} = \psi^\dagger\gamma^0 = (-\psi_-, \psi_+)$.

Ignoring potencial boundary terms (for now), the eom for ψ^μ_A is the DIRAC equation

$\gamma^a\partial_a\psi = 0$ or in components $\partial_+\psi_- = 0, \quad \partial_-\psi_+ = 0$

The symmetries of the susy action:

- Worldsheet reparametrization invariance.
- Space-time diffs.
- Weyl invariance.

And we have a new symmetry!

The action $S_B + S_F$ is invariant under:

$$\sqrt{\frac{2}{l_s^2}}\delta X^\mu = i\bar{\epsilon}\psi^\mu \quad \text{and} \quad \delta\psi^\mu = \sqrt{\frac{2}{l_s^2}}\frac{1}{2}\gamma^a\partial_a X^\mu \cdot \epsilon$$

donde $\epsilon_A = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}$ es un spinor infinitesimal de Majorana.

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In order for these to be a symmetry of the full action ϵ_A must obey

$$\gamma^b\gamma_a\partial_b\epsilon = 0$$

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This symmetry relates the bosonic and fermionic degrees of freedom.
This is the characteristic property of a supersymmetry (SUSY).

The symmetries of the susy action:

- → Worldsheet reparametrization invariance.
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- → Weyl invariance.

And we have a new symmetry!

SUSY is a deep concept that extends (in some sense uniquely) the Poincare symmetry. While found for the first time in the context of the two-dimensional RNS theory, it has become an important principle of more general physical systems.

We can now continue with our analysis of the flat gauge RNS action and proceed to the boundary conditions and mode expansion of the worldsheet fields.

→ The bosonic mode expansion and boundary conditions are just as in the bosonic string.

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The most general boundary conditions that do not mix ψ_+ and ψ_- and respect the space-time Poincaré symmetry are

$$\begin{cases} \psi_+^\mu(\sigma) = \pm\psi_+^\mu(\sigma + l) \\ \psi_-^\mu(\sigma) = \pm\psi_-^\mu(\sigma + l) \end{cases}$$

Since ψ is a worldsheet spinor, the minus sign is possible as we go around the worldsheet once, taking $\sigma \to \sigma + l$.

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The most general boundary conditions that do not mix ψ_+ and ψ_- and respect the space-time Poincaré symmetry are

$$\left\{ \begin{aligned} &\psi_+^\mu(\sigma) = \pm\psi_+^\mu(\sigma + l) \\ &\psi_-^\mu(\sigma) = \pm\psi_-^\mu(\sigma + l) \end{aligned} \right.$$

Notice that there are 4 independent sectors since for $\psi_\pm$ we can independently choose either sign.

In short, the boundary conditions can be written as

$$\psi_{\pm}(\sigma + l) = e^{2\pi i \triangle_{\pm}} \psi_{\pm}(\sigma) \quad \text{where} \quad \begin{cases} \triangle = 0 & \text{Ramond sector} \\ \triangle = \dfrac{1}{2} & \text{Neveu-Schwarz sector} \end{cases}$$

The Ramond sector (R) corresponds to periodic boundary conditions with integer mode expansion

$$\psi_{-}^{\mu}(\sigma, \tau) = \sum_{n \epsilon \mathbb{Z}} \sqrt{\frac{2\pi}{l}} b_n^{\mu} e^{-\frac{2\pi}{l} i n (\tau - \sigma)}$$

$$\psi_{+}^{\mu}(\sigma, \tau) = \sum_{n \epsilon \mathbb{Z}} \sqrt{\frac{2\pi}{l}} \tilde{b}_n^{\mu} e^{-\frac{2\pi}{l} i n (\tau + \sigma)}$$

In short, the boundary conditions can be written as

$$\psi_\pm(\sigma + l) = e^{2\pi i \triangle_\pm} \psi_\pm(\sigma) \quad \text{where} \quad \begin{cases} \triangle = 0 & \text{Ramond sector} \\ \triangle = \dfrac{1}{2} & \text{Neveu-Schwarz sector} \end{cases}$$

The Neveu-Schwarz sector (NS) corresponds to anti-periodic boundary conditions with half-integer mode expansion

$$\psi_-^\mu(\sigma, \tau) = \sum_{r \epsilon \mathbb{Z} + \frac{1}{2}} \sqrt{\frac{2\pi}{l}} b_r^\mu e^{-\frac{2\pi}{l} i r (\tau - \sigma)}$$

$$\psi_+^\mu(\sigma, \tau) = \sum_{r \epsilon \mathbb{Z} + \frac{1}{2}} \sqrt{\frac{2\pi}{l}} \tilde{b}_r^\mu e^{-\frac{2\pi}{l} i r (\tau + \sigma)}$$

In short, the boundary conditions can be written as

$$\psi_\pm(\sigma + l) = e^{2\pi i \triangle_\pm}\psi_\pm(\sigma)$$ where $$\begin{cases} \triangle = 0 & \text{Ramond sector} \\ \triangle = \dfrac{1}{2} & \text{Neveu-Schwarz sector} \end{cases}$$

The four different sectors are therefore

$$(\triangle_+, \triangle_-) = (0, 0) \longrightarrow \text{R-R}$$

$$(\triangle_+, \triangle_-) = (\frac{1}{2}, \frac{1}{2}) \longrightarrow \text{NS-NS}$$

$$(\triangle_+, \triangle_-) = (\frac{1}{2}, 0) \longrightarrow \text{NS-R}$$

$$(\triangle_+, \triangle_-) = (0, \frac{1}{2}) \longrightarrow \text{R-NS}$$

We want to quantize the theory so we postulate (anti) commutations relations:

- The X^μ -sector modes continue to enjoy the familiar commutation relations

- The fermions ψ_A^μ satisfy the canonical anti-commutation relations

$$\{\psi_+^\mu(\tau,\sigma),\psi_+^\nu(\tau,\sigma')\} = 2\pi\eta^{\mu\nu}\delta(\sigma-\sigma')$$

$$\{\psi_-^\mu(\tau,\sigma),\psi_-^\nu(\tau,\sigma')\} = 2\pi\eta^{\mu\nu}\delta(\sigma-\sigma')$$

$$\{\psi_+^\mu(\tau,\sigma),\psi_-^\nu(\tau,\sigma')\} = 0$$

Then

$$\{b_m^\mu, b_n^\nu\} = \{\tilde{b}_m^\mu, \tilde{b}_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0}$$

At this point we would've construct the Fock space and then use the corresponding (quantum) constraints to obtain the physical states.

Due to lack of time (and knowledge) we will jump to the end result! (this is a huge jump!).

Following similar arguments as we did for the bosonic string, one can show that consistency of the theory at the quantum level implies:

$\boxed{\text{10 space-time dimensions}}$

One can show that there are five consistent ways of combining the different sectors (then we have five “different” theories):

1. Type IIB theory the following four sectors are in the spectrum:

$$(\mathrm{NS}_+;\mathrm{NS}_+) \longrightarrow \Phi,\ B_{[\mu\nu]}\,,G_{(\mu\nu)}$$

$$(R_+;R_+) \longrightarrow C^{(0)},\ C^{(2)}_{[\mu_1\mu_2]}\,,C^{(4)}_{[\mu_1\mu_2\mu_3\mu_4]}$$

$$(\mathrm{NS}_+;R_+) \longrightarrow \lambda_a\,,\psi^\mu_a$$

$$(R_+;\mathrm{NS}_+) \longrightarrow \lambda_a\,,\psi^\mu_a$$

The theory is chiral because left- and right-movers have the same chirality.

One can show that there are five consistent ways of combining the different sectors (then we have five “different” theories):

1. Type IIA theory the following four sectors are in the spectrum:

$$(\mathrm{NS}_+;\mathrm{NS}_+) \longrightarrow \Phi,\ B_{[\mu\nu]}\ ,G_{(\mu\nu)}$$

$$(R_+;R_+) \longrightarrow C^{(1)}_{[\mu_1},\ C^{(3)}_{[\mu_1\mu_2\mu_2]}$$

$$(\mathrm{NS}_+;R_+) \longrightarrow \tilde{\lambda}_a\ ,\ \tilde{\psi}^{\mu}_a$$

$$(R_+;\mathrm{NS}_+) \longrightarrow \lambda_a\ ,\ \psi^{\mu}_a$$

Here left- and right-movers have opposite chirality

Type IIA and Type IIB both contain an equal number of bosonic and fermionic degrees of freedom, e.g. 128 + 128 at the massless level. This is a necessary condition for space-time supersymmetry , which exchanges bosonic and fermionic fields.

The 10-dimensional low-energy effective action keeping only the massless modes for Type IIA and Type IIB theory can be computed order by order in spacetime and worldsheet perturbation theory, by generalising the methods we got to know in the bosonic theory.

For example, the action for Type IIB takes the form

$$S = \frac{1}{(2\pi)^7 g_s^2 l_s^8} \int d^{10}x \sqrt{g} [R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}e^{-\phi}H_3^2 - \frac{1}{2}e^{2\phi}F_1^2 - \frac{1}{2}e^{\phi}F_3^2 - \frac{1}{4}F_5^2] - \frac{1}{(2\pi)^7 g_s^2 l_s^8} \int C_4 \wedge H_3 \wedge F_3,$$

We will not say anything about the other three cases …

(Just there names: Type I and Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$)

Final remark

Until 1995 it seemed that all these 5 consistent theories in 10 dimensions were independent. However, very smart people realised that they are related by dualities (Oscar's lectures). Thus they should be interpreted as different manifestations of one underlying theory.